Santosh Kumar Yadav

# Advanced Graph Theory

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# Preface

***"As long as a branch of science offers an abundance of problems, so long it is alive; a lack of problem foreshadows extinction or the cessation of independent development. Just as any human undertaking pursues certain objects, so also mathematical research requires its problems. It is by the solution of problems that the investigator tests the temper of his steel; he finds new methods and new outlooks, and gains a wider and freer horizon."***

These words of **David Hilbert** impressed me and motivated to write this book as a counter part of mathematics. There are several reasons for acceleration of interest in Graph theory. While Graph theory is intimately related to many branches of mathematical sciences, it also has applications to some areas of physical, chemical, communication, computer, engineering and social science. In fact, it serves as a mathematical model for any system involving a binary relation. Partly, because of their diagrammatic representation, graphs have an intuitive and aesthetic appeal. While many results in graph theory are of elementary nature, there is abundance of problems with enough combinatorial subtlety to challenge the most sophisticated mathematics.

The present book is based on the curriculum of undergraduate and postgraduate courses of universities in India and abroad. Every effort has been made to present the various topics in the theory of graphs in a logical manner with adequate historical background and include suitable figures to illustrate concepts and results ideally. The formidable exercises, neither easy nor straight forward, are bold faced and highlighted. The theory portion of each chapter must be studied thoroughly as it would help solve many of the problems with comparative ease. The instructor may select material from this book for a semester course on graph theory, while the entire book can serve for a whole session course.

**Dr. Santosh Kumar Yadav**

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# Acknowledgments

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Lastly I want to thank my readers who have always encouraged me and appreciated my work. I must accept responsibility, for any errors that may still remain. I would certainly appreciate receiving comments about any unwelcome surprises, alternate or better solutions, exercises, puzzles and applications.

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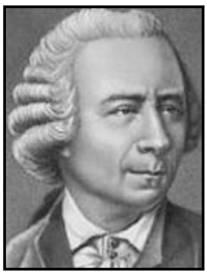
# 1 Basics of Graph Theory



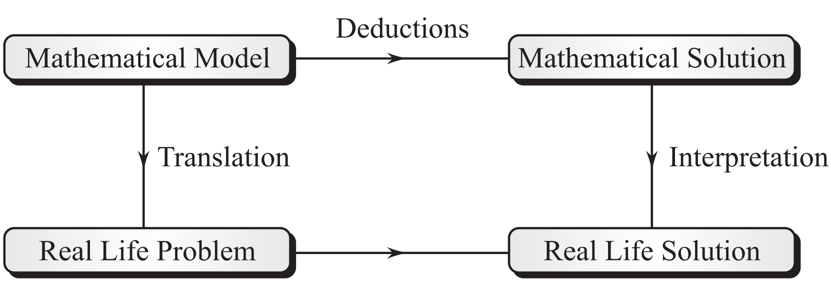
**Leonhard Euler (1707-1783)**

Leonhard Euler (1707-1783) was born in an intellectual family in Basel, Switzerland. His father was a mathematician and a calvinist paster and wanted him to be a pastor in future. Euler had different ideas, he followed his father's wishes and entered in the university of Basel to study Hebrew and theology. For his hard work and remarkable ability the famous mathematician Johann Bernoulli was very much impressed and recognized this young chap's talent. He approached Euler's father to change the track and got a positive response. Euler was allowed to pursue his studies in mathematics.

Euler brought out his first paper. He won Paris Academy prize 12 times. He was the most prolific mathematician and contributed every branch of mathematics. With his phenominal memory, he had perfect recall for every result. As a genius, he could work anywhere and under any conditions. Euler belongs in a class by himself. Euler is known as the father of Graph theory.

## 1.1 Introduction

In the last five decades graph theory has established itself as a worthwhile mathematical discipline and there are many applications of graph theory to a wide variety of subjects that include Operations Research, Physical Sciences, Economics, Genetics, Sociology, Linguistics, Engineering and Computer Science etc. Such a development may be roughly described as follows





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Graph theory has been independently discovered many times through some puzzles that arose from the physical world, consideration of chemical isomers, electrical networks etc. The graph theory has been independently discovered many times, since it may be considered as a counter part of Applied Mathematics.

> In the words of Sylvester, *"the theory of ramification is one of pure colligation, for it takes no account of magnitude or position, geometrical lines are used, but have no more real bearing on the matter than those employed in genealogical tables have in explaining the laws of procreation."*

In the present century, there have already been as great many rediscoveries of graph theory which we can only mention most briefly in this chronological account.

Graph theory is considered to have begun in 1736 with the publication of Euler's solution of the Königsberg bridge problem. Euler (1707–1782) is known as the father of graph theory as well as topology.

There are various types of graphs, each with its own definition. Unfortunately, some people apply the term "graph" rather loosely, so we cannot be sure what type of graph we are talking about unless we ask them. After we have finished this chapter, we expect us to use the terminology carefully not loosely. To motivate the various definitions we will take suitable example.

## 1.2 Graph! What is it?

### 1.2.1 *Simple Graph*

A simple graph $G$ is a pair $G = (V, E)$

Where

- $V$ is a finite set, called the vertices of $G$
- $E$ is a subset of $P_2(V)$ (a set $E$ of two-element subset of $V$), called the edges of $G$.

To avoid rotational ambiguities, we always assume tactly that $V \cap E = \phi$.

A nuisance in first learning graph theory is that there are so many definitions. They all correspond to intuitive ideas, but can take a while to absorb. Some ideas have multiple names. For example, graphs are sometimes called **networks**, vertices are sometimes called **nodes**, and edges are sometimes called **areas**. Even worse, no one can agree on the exact meanings of these terms. For example, in our definition, every graph must have at least one vertex. However, other authors permit graphs with no vertices. (The graph with no vertices is the single, stupid counter example to many would–be theorems–so we're banning it!). This is typical; everyone agrees more-or-less what each terms means, but disagrees

about weird special cases. So do not be alarmed if definitions here differ subtly from definitions we see elsewhere. Usually, these differences do not matter.

If we take an example of graph as shown in Fig. 1.1

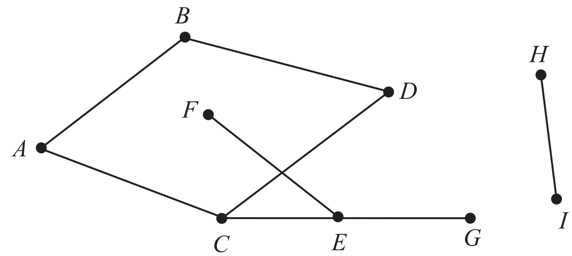



**Fig. 1.1: Simple Graph**

The vertices correspond to the dots in the picture, and the edges correspond to the lines. Thus, the dots and lines diagram above in a pictorial representation of the graph $(V, E)$ where

$V = \{A, B, C, D, E, F, G, H, I\}$

$E = \{\{A, B\}, \{A, C\}, \{B, D\}, \{C, D\}, \{C, E\}, \{E, F\}, \{E, G\}, \{H, I\}\}$

Hereafter, we use $A—B$ to denote an edge between vertices A and B rather than the set notation $\{A, B\}$. Note that $A—B$ and $B—A$ are the same edge, just as $\{A, B\}$ and $\{B, A\}$ are the same set.

Two vertices in a graph are said to be adjacent if they are joined by an edge, and an edge is said to be incident to the vertices it joins. The number of edges incident to a vertex is called the degree of the vertex. For example, in the graph above, $A$ is adjacent to $B$ and $B$ is adjacent to $D$, and the edge $A—C$ is incident to vertices $A$ and $C$. Vertex $H$ has degree 1, $D$ has degree 2, and $E$ has degree 3.

Deleting some vertices or edges from a graph leaves a subgraph. Formally, a subgraph of $G = (V, E)$ is a graph $G' = (V', E')$ where $V'$ is a nonempty subset of $V$ and $E'$ is a subset of $E$. Since a subgraph is itself a graph, the end points of every edge in $E$' must be vertices in $V'$.

■ **Example 1.1:** *A Computer Network Problem*

Computers are linked with one another so that they can interchange information through a particular network. Given a collection of computers, we can describe this linkage in clear terms so that we might answer the questions like; "How can we send a message from computer $A$ to computer $B$ using the fewest possible intermediate computers?"

We can do this by making a list coasting of pairs of computer that are interconnected. The pairs are unordered since, if computer $C$ can communicate

with computer $D$, then the reverse is also true. We have implicitly assumed that the computers are distinguished from each other. It is sufficient to say that "*A PC* is connected to Ayan." We must specify which *PC* and which Ayan. Thus, each computer has a unique identification label of some sort.

For those who like pictures rather than lists, we can put dots on a piece of paper, one for each computer we label each dot with a computer's identifying label and draw a curve connecting two dots if the corresponding computers are connected. The shape of the curve does not matter because we are only interested in whether two computers are connected or not. We can see two such pictures of the same graph (as shown in Fig. 1.2). Each computer has been labeled by initials of its owner.

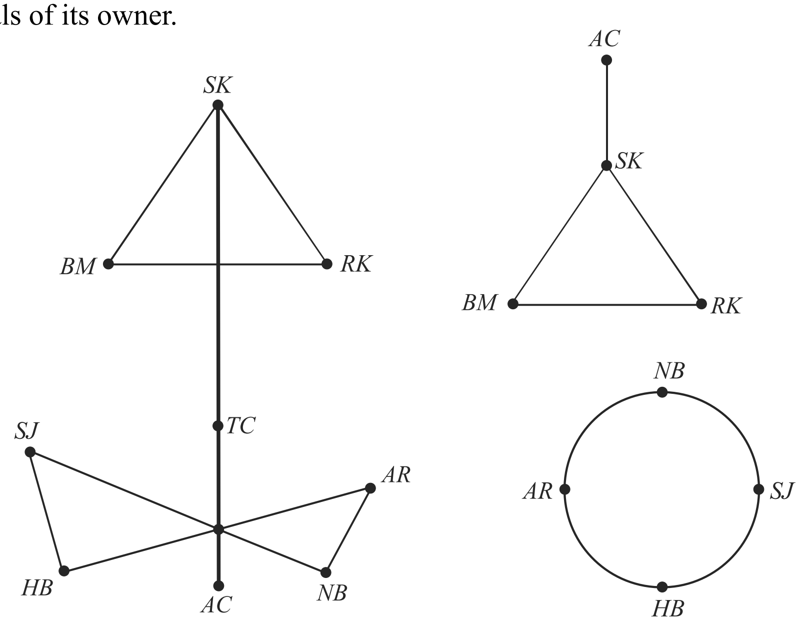



**Fig. 1.2: Computer Network Problem**

Computer (vertices) are indicated by dots (·) with labels. The connections (edges) are indicated by lines. When limes cross, they might be thought of as cables that live on top of each other not as cables that are joined.

### 1.2.2 *Graph*

*A graph is a triple* $G = (V, E, \phi)$

Where

- $V$ *is a finite set, called the vertices of* $G$.
- $E$ *is a finite set, called the edges of* $G$
- $\phi$ *is a function with domain* $E$ *and Co-domain* $P_2(V)$.

This definition tells us that to specify a graph $G$ it is necessary to specify the sets $V$ and $E$ and the function $\phi$. We have to specify $V$ and $\phi$ in set theoretic terms.

The function $\phi$ is sometimes called the incidence function of the graph. The two element of $\phi(x) = \{u, v\}$, for any $x \in E$, are called the vertices of the edge $x$, and we say $u$ and $v$ are joined by $x$. We can also say that $u$ and $v$ are adjacent vertices and $u$ adjacent to $v$ or, equivalently, $v$ is adjacent to $u$.

For any $u \in v$, if $v$ is a vertex of an edge $x$ than we can say the $x$ is incident on $v$. Likewise, we can say $v$ is the member of $x$, $v$ is on $x$, or $v$ is in $x$. Of course, $v$ is a member of $x$ actually means $v$ is a member of $\phi(x)$.

■ **Example 1.2:** *Problem of Routes between Cities (Geographical Placement Problem)*

We consider four cities named, (with characteristic mathematical charm), $A$, $B$, $C$ and $D$. Between these cities there are various routers of travel, denoted by $a$, $b$, $c$, $d$, $e$, $f$ and $g$ (as shown in Fig. 1.3).

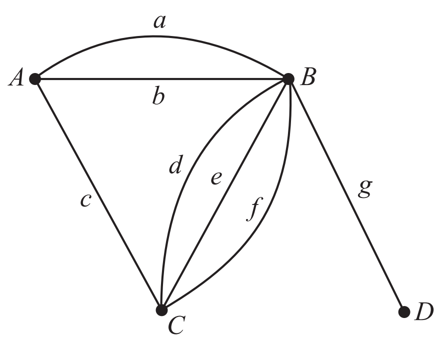



**Fig. 1.3: Geographical Placement Problem**

On observing this picture, we can see that there are three roots between cities $B$ and $C$. These routs are $d$, $e$ and $f$. This picture is intended to give us only information about the interconnections between the cities. This leaves out so many aspects of the situation that might be an interest to the traveler. Unlike a typical map, no claim is made that the picture represents in any way the distance between the cities or their geographical placement relative to each other. The object shown in this picture is a graph.

One is tempted to list the pairs of the cities which are connected; in other words, to extent a simple graph from the information. This does not describe the problem adequately because there can be more than one route connecting a pair of cities *e.g.* $d$, $e$ and $f$ connecting cities $B$ and $C$ as shown in Fig. 1.3.

In the pictorial representation of the cities graph

$G = (V, E, \phi)$ where

$V = \{A, B, C, D\}$, $E = \{a, b, c, d, e, f, g\}$.

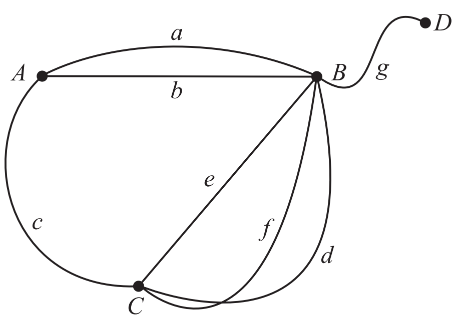
and

$$\phi = \begin{pmatrix} a & b & c & d & e & f & g \\ \{A, B\} & \{A, B\} & \{A, C\} & \{B, C\} & \{B, C\} & \{B, C\} & \{B, D\} \end{pmatrix}$$

The function $\phi$ is determined from the picture by comparing the name attached to a route with the two cities connected by that route. Thus, the route name $d$ is attached to the route with end points $B$ and $C$. This means that $\phi(d) = \{B, C\}$

Since part of the definition of a function includes its co-domain and domain, $\phi$ determined $P_2(V)$ and $E$. Also, $V$ can be determined from $P_2(V)$. Consequently, we could have said a graph is a function $\phi$ whose domain is a finite set and co-domain is $P_2(V)$ for some finite set $V$. Instead, we choose to specify $V$ and $E$ explicitly because the vertices and edges play a functional role in thinking about graph $G$.

Fig 1.4 Shows two additional pictures of the same cities graph given above (Fig. 1.3)



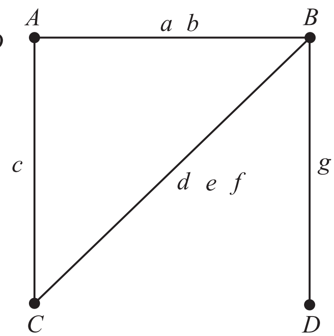




**Fig. 1.4 Same Cities Location Graph**

The drawings look very different but exactly the same set $V$ and function $\phi$ are specified in each case. It is very important that are to understand exactly what information is needed to completely specify the graph. When thinking in terms of cities and routes between them, we want the pictorial representation of the cities to represent their geographical positioning. If the pictorial representation does this, that's fine, but it is not a part of the information required to define a graph. The geometrical positioning of the vertices, $A$, $B$, $C$ and $D$ is very different, in the first of the two pictorial representations above, than it was in our original representation of the cities. However, in each of these cases, the vertices on a given edge are the same and hence the graphs specified are the same. In the second of two pictures above, a different method of specifying the graph is given. There, $\phi^{-1}$ the inverse of $\phi$, is given. For example, $\phi^{-1}(\{C, B\})$ is shown to be $\{d, e, f\}$. Knowing $\phi^{-1}$ determines $\phi$ and hence determines $G$ since the vertices $A$, $B$, $C$ and $D$ are also specified.

### 1.2.3 Loops

**A loop is an edge which connects a vertex to itself**. Graphs and simple graphs cannot have loops, why? Let us explain it. Suppose $e \in$ E is a loop in a graph that connects $v \in V$ to itself. Then $\phi(e) = \{u, v\} = \{v\}$ because repeated elements in the description of a set count only once. They are the same element.

Since $\{v\} \notin P_2(V)$, the range of $\phi$, we cannot have $\phi(e) = \{u, v\}$ *i.e.* we cannot have a loop.

If we want to allow loops, we will have to change our definitions. For a graph, we expand the codomain of $\phi$ to be $P_2(V) \cup P_1(V)$. For a simple graph we need to change the set of allowed edges to include loops. This could be done by saying that $E$ is a subset of $P_2(V) \cup P_1(V)$ instead of a subset of just $P_2(V)$. For example, if $V = \{1, 2, 3\}$ and $E = [\{1, 2\}, \{2\}, \{2, 3\}]$, this simple graph has a loop at vertex 2 and vertex 2 is connected by edge to the other two vertices. When we want to allow loops, we can say about a graph with loops on a simple graph with loops.

Now we correlate simple graph and graph. Let $G = (V, E)$ be a simple graph. Define $\phi : E \rightarrow E$ to be identity map; *i.e.* $\phi(e) = e$ for all $e \in E$.

The graph $G' = (V, E, \phi)$ is essentially the same as $G$. There is one subtle difference in the picture: The edges of $G$ are unlabeled but each edge of $G'$ is labeled by a set consisting of the two vertices at its ends. But this extra information is contained already in the specification of $G$. Thus, we can say that simple graphs are special case of graphs.

### 1.2.4 Degree of Vertices

Let $G = (V, E, \phi)$ be a graph and $v \in V$ a vertex. Define the degree of $v$, $d(v)$ to be the number of $e \in E$ such that $v \in \phi(e)$; i.e; $e$ is incident on $v$ suppose $|v| = n$. Let $d_1, d_2, \ldots, d_n$, where $d_1 \leq d_2 \leq, \ldots, \leq d_n$ be the sequence of degrees of the vertices of $G$, sorted by size. We refer to this sequence as the degree sequence of the graph $G$.

From Fig. 1.4 (the graph for routes between cities) $d(A) = 3$, $d(B) = 6$, $d(C) = 4$ and $d(D) = 1$. The degree sequence is 1, 3, 4, 6.

#### *Theorem 1.1*

The number of vertices of odd degree in a graph is always even.

#### *Proof:*

Let G be a graph of size $m$.

We divide $V(G)$ into two subsets $V_1$ and $V_2$ where $V_1$ consists of the odd vertices of $G$ and $V_2$ consists of the even vertices of $G$.

Since, if G is a graph of size $m$, then

$$\sum_{v \in V(G)} \deg v = 2m \qquad \text{... } (i)$$

We can conclude by equ. ($i$) that,

$$\sum_{v \in V(G)} \deg v = \sum_{v \in V_1} \deg v + \sum_{v \in V_2} \deg v = 2m \qquad \text{... } (ii)$$

The number $\sum_{v \in V_2} \deg v$ is even since it is a sum of even integers.

Thus

$$\sum_{v \in V_1} \deg v = 2m - \sum_{v \in V_2} \deg v \qquad \text{... } (iii)$$

The above implies that

$\sum_{v \in V_1} \deg v$ is even

Since each of the number deg $v$, $v \in V_1$ is odd the number of odd vertices of $G$ is even. ■

***Note:*** *A graph in which all vertices are of equal degree is called a regular graph.*

In order to have a partition of $S$, we must have

($a$) the $B(s)$ are nonempty and every $t \in S$ is in some $B(s)$ and

($b$) for every $p, q \in S$, $B(p)$ and $B(q)$ are either equal or disjoint.

Since ~ is reflexive, $s \in B(s)$, proving ($a$). Suppose $x \in B(p) \cap B(q)$ and $y \in B(p)$. We have, $p \sim x$, $q \sim x$ and $p \sim y$. Thus $q \sim x \sim p \sim y$ and so $y \in B(q)$, proving that $B(p) \subseteq B(q)$. Similarly $B(q) \subseteq B(p)$ and so $B(p) = B(q)$. This proves ($b$).

■ **Example 1.3:** *Sex in America (Matching Problem)*

A 1994 University of Chicago study entitled ***The social Organization of Sexuality*** found that on average men have 74% more opposite-gender partners than women.

Let us recast this observation in graph theoretic terms. Let $G = (V, E)$ be a graph where the set of vertices $V$ consists of everyone in America. Now each vertex either represents either a man or a woman, so we can partition $V$ into two subsets: $M$, which contains all the male vertices, and $W$, which contains all the

female vertices. Let's draw all the $M$ vertices on the left and the $W$ vertices on the right as shown in Fig. 1.5.

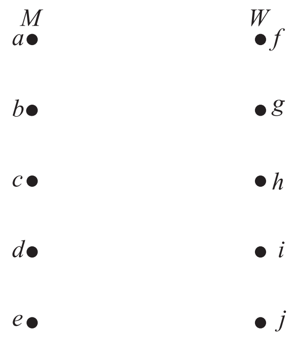



**Fig. 1.5: Matching Problem**

Now, without getting into a lot of specifics, sometimes an edge appears between an $M$ vertex and a $W$ vertex. (Fig. 1.6)

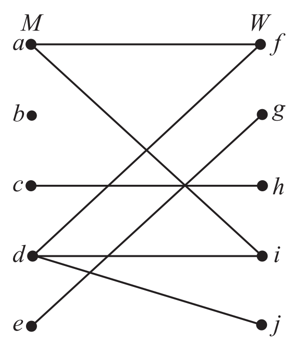



**Fig. 1.6: Perfect Match**

Since we're only considering opposite-gender relationships, every edge connects an $M$ vertex on the left to a $W$ vertex on the right. So the sum of degrees of the $M$ vertices must equal the sum of the degrees of the $W$ vertices:

$$\sum_{x \in M} \deg(x) = \sum_{y \in W} \deg(y)$$

Now suppose we divide both sides of this equation by the product of the sizes of the two sets, $|M|.|W|$:

$$\left(\frac{\sum_{x \in M} \deg(x)}{|M|}\right) \cdot \frac{1}{|W|} = \left(\frac{\sum_{y \in W} \deg(y)}{|W|}\right) \cdot \frac{1}{|M|}$$

The terms above in parentheses are the average degree of an $M$ vertex and the average degree of a $W$ vertex. So we know:

$$\frac{\text{Avg. deg in } M}{|W|} = \frac{\text{Avg. deg in } W}{|M|}$$

$$\text{Avg. deg in M} = \frac{|W|}{|M|}.\ \text{Avg. deg in } W.$$

Now the Census Bureau reports that there are slightly more women than men in America; in particular $|W| / |M|$ is about 1.035. So– assuming the Census Bureau is correct– we have just proved that the University of Chicago study got bad data! On average, men have 3.5% more opposite-gender partners. Furthermore, this is totally unaffected by differences in sexual practices between men and women; rather, it is completely determined by the relative number of men and women!

We consider a graph $G$ with $e$ edges and $n$ vertices $v_1, v_2, \ldots v_n$. Since each edge contributes two degrees, the sum of the degrees of all vertices in $G$ is twice the number of edges in $G$. That is

$$\sum_{i=1}^{n} d(v_i) = 2e \qquad \ldots(i)$$

From Fig. 1.4. $d(A) + d(B) + d(C) + d(D) = 3 + 6 + 4 + 1 = 14$ *i.e.* twice the number of edges.

### 1.2.5 Equivalence Relation

An equivalence relation on a set $S$ is a partition of $S$. We say that $s, t \in$ S are equivalent if and only if they belong to the same block (called an equivalence class in this context) of the partition. If the symbol ~ denotes the equivalence relation, then we write s ~ t to indicate that s and t are equivalent.

■ **Example 1.4:** *Illustration of Equivalence Relations*

Let $S$ be any set and let all the blocks of the partition have one element. Two elements of $S$ are equivalent if and only if they are the same. This rather trivial equivalence relation is, of course, denoted by "=".

Now let the set be the integers $Z$. Let's try to define an equivalence relation by saying that $n$ and $k$ are equivalent if and only if they differ by a multiple of 24. Is this an equivalence relation? If it is, we should be able to find the blocks of the partition. There are 24 of them, which we call number 0, …, 23. Block $j$ consists of all integers which equals $j$ plus a multiple of 24; that is, they have a remainder of $j$ when divided by 24. Since two numbers belong to the same

block if and only if they both have the same remainder when divided by 24, it follows that they belong to the same block if and only if their difference gives a remainder of 0 when divided by 24, which is the same as saying their difference is a multiple of 24. Thus this partition does indeed give the desired equivalence relation.

Now let the set be $Z \times Z^*$. where $Z^*$ is the set of all integers except $O$. Write $(a, b) \sim (c, d)$ if and only if $ad = bc$. With a moment's reflection, we should see that this is a way to check if the two fractions $a/b$ and $c/d$ are equal. We can label each equivalence class with the fraction $a/b$ that it represents. In an axiomatic development of the rationals from the integers, we define a rational number to be just such an equivalence class and proves that it is possible to add, subtract, multiply and divide equivalence classes.

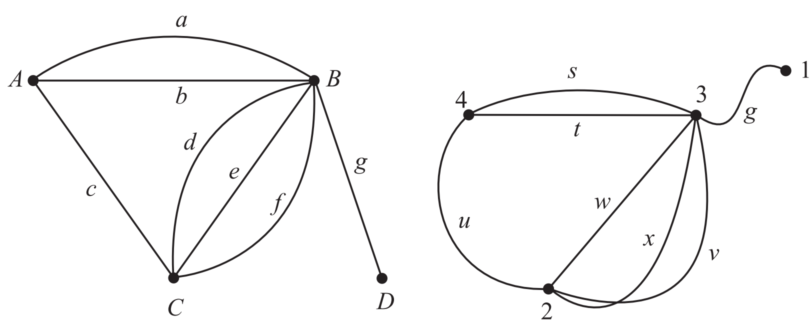
Suppose we consider all functions $S = \underline{m}^{\underline{n}}$. We can define a partition of $S$ in a number of different ways. For example, we could partition the functions $f$ into blocks where the sum of the integers in the Image $(f)$ is constant, where the max of the integers in Image $(f)$ is constant, or where the "type vector" of the function, namely, the number of 1's, 2's, etc. in Image$(f)$, is constant. Each of these defines a partition of $S$. ■

■ **Example 1.5:** *Ghosts of Departed Graphs*

We consider the following two graphs represented by Fig 1.7.

Let us remove all symbols representing edges and vertices. We have left two "forms" on which the graphs were drawn, we can think of drawing a picture of a graph as a two step process

($i$) draw the form ($ii$) add the labels




$G_a$ $G_b$

**Fig. 1.7: Ghosts of Departed Graphs**

We get form $F_a$ and Form $F_b$ have a certain similarity as shown in Fig 1.8.

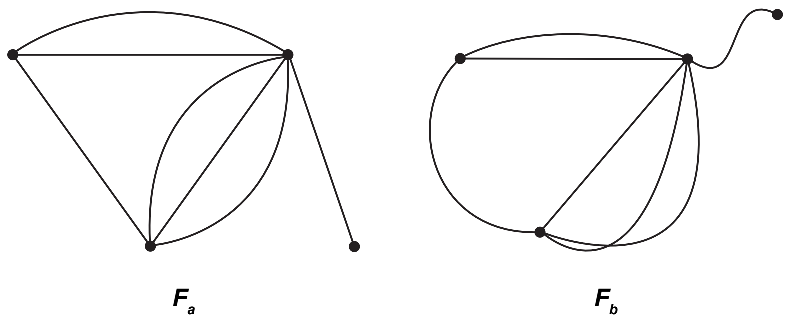



**Fig. 1.8: Similarity of Graphs**

Now suppose we have any two graphs, $G_a = (V_a, E_a, \phi_b)$. We can think of these graphs not as pictures, but as specified in terms of sets and functions. Now we choose forms $F_a$ and $F_b$ for $G_a$ and $G_b$ respectively, and draw their pictures. We leave it to our intuition to accept the fact that either $F_a \sim F_b$, no matter what we choose for $F_a$ and $F_b$, or $F_a \nsim \mathrm{F_b}$ no matter what our choice is for the forms $F_a$ and $F_b$. If $F_a \sim F_b$ we say that $G_a$ and $G_b$ are isomorphic graphs and write $G_a \approx G_b$. The fact that ~ is an equivalence relation forces ≈ to be an equivalence relation also. In particular, two graphs $G_a$ and use that same form for $G_b$. ■

### *Theorem 1.2*

Let $S$ be a set and suppose that we have a binary relation $R \subseteq S \times S$. We write $s \sim t$ whenever $(s, t) \in R$. This is an equivalence relation if and only if the following three conditions hold.

(*i*) (Reflexive) For all $s \in S$ we have $s \sim s$.

(*ii*) (Symmetric) For all $s, t \in S$ such that $s \sim t$ we have $t \sim s$.

(*iii*) (Transitive) For all $r, s, t \in S$ such that $r \sim s$ and $s \sim t$ we have $r \sim t$.

### *Proof:*

We first prove that an equivalence relation satisfies (i)-(iii). Suppose that ~ is an equivalence relation. Since $s$ belongs to whatever block it is in, we have $s \sim s$. Since $s \sim t$ means that $s$ and $t$ belong to the same block, we have $s \sim t$ if and only if we have $t \sim s$. Now suppose that $r \sim s \sim t$. Then $r$ and $s$ are in the same block and $s$ and $t$ are in the same block. Thus $r$ and $t$ are in the same block and so $r \sim t$.

We now suppose that (i)-(iii) hold and prove that we have an equivalence relation. Everything equivalent to a given element should be in the same block. Thus, for each $s \in S$ let $B(s)$ be the set of all $t \in S$ such that $s \sim t$. We must show that the set of these sets from a partition of $S$.

■

### 1.2.6 Random Graph Model

Let $G$ ($n$, $p$) be the probability space obtained by letting the elementary events be the set of all $n$-vertex simple graphs with $V = \underline{n}$. If $G \in G$ ($n$, $p$) has $m$ edges, the $P(G) = p^m q^{N-m}$ where $q = 1 - p$ and $N = \binom{n}{2}$

We need to show that $G$ ($n$, $p$) is a probability space. There is a nice way to see this by reinterpreting $P$. List the $N = \binom{n}{2}$ vertices $P_2(V)$ in lex order. Let the sample space be $U = \times^N$ {choose, reject} with $P(a_1,...,a_1,...,a_N) = P^*(a_1) \times ... \times P^*(a_N)$ where $P^*$ (choose) $= p$ and $P^*$ (reject) $= 1 - p$. We've met this before in Unit $F_n$ and seen that it is a probability space. To see that it is, note that $P \geq 0$ and

$$\sum_{a_1,...,a_N} P(a_1,...,a_N) = \sum_{a_1,...,a_N} P^*(a_1)\times,\ldots,\times P^*(a_N)$$

$$= \left(\sum_{a_1} P^*(a_1)\right)\times,\ldots,\times\left(\sum_{a_N} P^*(a_N)\right)$$

$$= (p + (1 - p)) \times ,\ldots, \times (p + (1 - p))$$

$$= 1 \times ,\ldots, \times 1 = 1.$$

We think of the chosen pairs as the edges of a graph chosen randomly from $G(n, p)$. If $G$ has $m$ edges, then its probability should be $p^m(1 - p)^{N-m}$ according to the definition. On the other hand, since $G$ has $m$ edges, exactly $m$ of $a_1$, ..., $a_N$ equal "choose" and so, in the new space, $P(a_1,..., a_N) = p^m (1 - P)^{N-m}$ also. We say that we are choosing the edges of the random graph independently.

### 1.2.7 Isolated Vertex, Pendent Vertex and Null Graph

A vertex having no incident edge is called an isolated vertex or isolated vertices are the vertices with zero degree.

In Fig. 1.9 vertices $v_4$ and $v_7$ are isolated vertices.

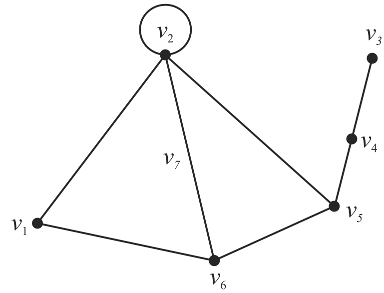



**Fig. 1.9: Isolated Vertex**

A vertex of degree one is called a pendent vertex or an end vertex. Vertex in Fig. 1.9 is a pendent vertex. Two adjacent edges are said to be in series if their common vertex is of degree two. In Fig. 1.9, the two edges incident on $v_1$ are in series.

***Note:*** *A graph without any edges, is called a null graph.*

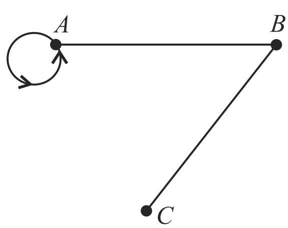
## 1.3 Digraphs

A directed graph (or digraph) is a triple $D = (V, E, \phi)$ where $V$ and $E$ are finite sets and $\phi$ is a function with domain $E$ and codomain $V \times V$. We call E the set of edges of the digraph $D$ and call $V$ the set of vertices of $D$.

We can define a notion of a simple digraph. A simple digraph is a pair $D = (V, E)$, where $V$ is a set, the vertex set, and $E\ x \subseteq V \times V$ is the edge set. Just as with simple graphs and graphs, simple digraphs are a special case of digraphs in which $\phi$ is the identify function on $E$; *i.e.* $Q(e) = e\ \forall\ e \in E$.

There is a correspondence between simple graphs and simple digraphs that is fairly common in application of graph theory. To interpret simple graphs in terms of simple digraphs, it is best to consider simple graphs with loops.

We consider $G = (V, E)$ where $E \subseteq P_2(V) \cup P_1(V)$. We may identify $\{u, v\} \in P_2(V) \cup P_1(V)$ with $(u, v) \in V \times V$ and with $(v, u) \in V \times V$. We identify $\{4\}$ with $(u, v)$. The Fig. 1.10 shows the simple graph and corresponding digraph.



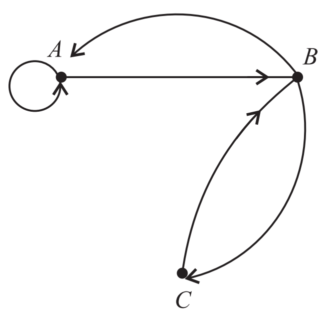




**Fig. 1.10: Show the Simple Graph and Corresponding Digraph**

Each, edge that is not a loop in the simple graph is replaced by two edges "in opposite directions" in the corresponding simple digraph. A loop is replaced by a directed loop (*e.g.*, $\{A\}$ is replaced by $(A, A)$).

Simple digraphs appear in mathematics under another important guise: binary relations. A binary relation on a set $V$ is simply a subset of $V \times V$. Often

the name of the relation and the subset are the same. Thus we speak of the binary relation $E \subseteq V \times V$. If you have absorbed all the terminology, you should be able to see immediately that $(V, E)$ is a simple digraph and that any simple digraph $(V' \times V')$ correspondence to a binary relation $E' \subseteq V' \times V'$.

We can recall that a binary relation $R$ is called symmetric if $(u, v) \in R$ implies $(v, u) \in R$. Thus simple graphs with loops correspond to symmetric binary relations.

An equivalence relation on a set $S$ is a particular type of binary relation $R \subseteq S \times S$. For an equivalence relation, we have $(x, y) \in R$ if and only if $x$ and $y$ are equivalent (*i.e.*, belong to the same equivalence class or block). Note that this is a symmetric relationship, so we may regard the associated simple digraph as a simple graph. Which simple graphs (with loops allowed) correspond to equivalence relations? As an example, take $S = \underline{7}$ and take the equivalence class partition to be $\{\{1, 2, 3, 4\}, \{5, 6, 7\}\}$. Since everything in each block is related to everything else, there are $\binom{4}{2} = 6$ non-loops and $\binom{4}{1} = 4$ loops associated with the block $\{1, 2, 3, 4\}$ for a total of ten edges. With the block $\{5, 6, 7\}$ there are three loops and three non-loops for a total of six edges. Here is the graph of this equivalence relation: (Fig. 1.11)

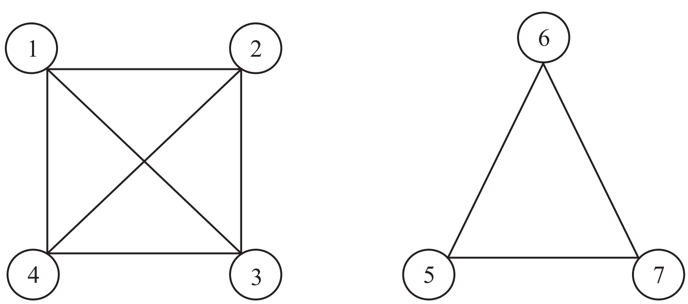



**Fig. 1.11: Graph of Equivalence Relation**

A complete simple graph $G = (V, E)$ with loops is a graph with every possible edge. That is, $E = P_2(V) \cup P_1(V)$. In the above graph, each block of the equivalence relation is replaced by the complete simple graph with loops on that block.

***Handshaking Lemma:*** For a graph $G$ we have

$$\sum_{u \in V(G)} d_G(u) = 2|E(G)|$$

### *Proof*

We consider a collection of guests in a partly.

Let some guests shook hands with some other guests.

If we asked everyone in that partly how many guests they shook hands with and added those numbers all up, this sum could be equal to twice the number of total hands shake.

In graph theoretic terms, each vertex represents a guest, and an edge between two guests represents a hand shake between them.

Each edge in E(G) has end vertices (say $u$ and $v$) which could be identical (if the edge is a loop), and hence contributes two to the sum in the left: One to $d_G(u)$ and other to $d_G(v)$ or two to $d_G(u)$ (if the edge is a loop).

The following two corollaries are possible as a consequence of the Hand-shaking theorem:

(*i*) In any graph there is always as even number of vertices having odd degree. i.e. the value $\sum_{u \in V(G)} d_G(u)$ is always even, and hence the number of odd $dG(u)$ terms in even i.e. the number of vertices of odd degree is even.

(*ii*) Every $k$-regular graph on $n$ vertices has $kn/2$ edges. The complete graph $K_n$ has $(n-2)n/2$ edges. ■

■ **Example 1.6:** *Flow of Commodities Problem*

We can recall example 2. Imagine now that the symbols $a$, $b$, $c$, $d$, $e$, $f$ and $g$, instead of standing for route names, stand for commodities (applesauce, bread, computers, etc.) that are produced in one town and shipped to another town. In order to get a picture of the flow of commodities, we need to know the directions in which they are shipped. This information is provided by picture below (Fig. 1.12)

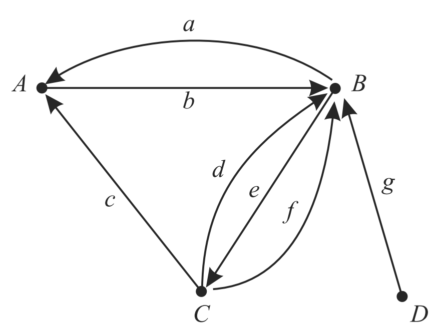



**Fig. 1.12: Flow of Commodities Problem**

In set-theoretic terms, the information needed to construct the above picture can be specified by giving a pair $D = (V, E, \phi)$ where $\phi$ is a function. The domain of the function $\phi$ is $E = \{a, b, c, d, e, f, g\}$ and the codomain is $V \times V$. Specifically,

$$\phi = \begin{pmatrix} a & b & c & d & e & f & g \\ (B,A) & (A,B) & (C,A) & (C,B) & (B,C) & (C,B) & (D,B) \end{pmatrix} \qquad \blacksquare$$

## 1.4 Path, Trail, Walk and Vertex Sequence

Let $G = (V, E, \phi)$ be a graph. Let $e_1, e_2,...,e_{n-1}$ be a sequence of elements of $E$ (edges of $G$) for which there is a sequence $a_1, a_2,...,a_n$ of distinct elements of $V$ (vertices of $G$) such that $\phi(e_i) = \{a_i, a_{i+1}\}$ for $i = 1, 2,..., n - 1$. The sequence of edges $e_1, e_2,...,e_{n-1}$ is called a path in $G$. The sequence of vertices $a_1, a_2,...,a_n$ is called the vertex sequence of the path. (Note that since the vertices are distinct, so are the edges.)

If we require that $e_1,...,e_{n-1}$ be distinct, but not that $a_1,...,a_n$ be distinct, the sequence of edges is called a *trail*.

If we do not even require that the edges be distinct, it is called a *walk*.

If $G = (V, E, \phi)$ is a directed graph, then $\phi(e_i) = \{a_i, a_{i+1}\}$ is replaced by $\phi(e_i) = (a_i, a_{i+1})$ in the above definition to obtain a directed path, trail, and walk respectively.

**Note:** *Every path is a trail and every trail is a walk, but not conversely.*

### *Theorem 1.3*

*Walk implies path*

Suppose $u \neq v$ are the vertices is a graph $G = (V, E, \phi)$.

The following are equivalent:

(*i*) There is a walk from $u$ to $v$.

(*ii*) There is a trail from $u$ to $v$.

(*iii*) There is a path from $u$ to $v$.

Furthermore, given a walk from $u$ to $v$, there is a path from $u$ to $v$ all of whose edges are in the walk.

### *Proof:*

Since every path is a trail, (*iii*) implies (*ii*). Since every trail is a walk, (*ii*) implies (*i*). Thus it suffices to prove that (*i*) implies (*iii*). Let $e_1, e_2,..., e_k$ be a walk from $u$ to $v$. We use induction on $n$, the number of repeated vertices in a walk. If the walk has no repeated vertices, it is a path. This starts the induction at $n = 0$. Suppose $n > 0$. Let $r$ be a repeated vertex. Suppose it first appears in edge $e_i$ and last appears in edge $e_j$.

If $r = u$, then $e_j,...,e_k$ is a walk from $u$ to $v$ in which $r$ is not a repeated vertex. If $r = v$, then $e_1,...,e_i$ is a walk from $u$ to $v$ in which $r$ is not a repeated vertex. Otherwise, $e_1,...,e_i$, $e_j,...,e_k$ is a walk from $u$ to $v$ in which $r$ is not a repeated vertex. Hence there are less than $n$ repeated vertices in this walk from $u$ to $v$ and so there is a path by induction. Since we constructed the path by removing edges from the walk, the last statement in the theorem follows.

■

## 1.5 Subgraph

Let $G = (V, E, \phi)$ be a graph. A graph $G' = (V', E', \phi')$ is a subgraph of $G$ iff $V' \subseteq V$, $E' \subseteq E$, and $\phi'$ is the restriction of $\phi$ to $E'$.

The fact that $G'$ is itself a graph means that $\phi(x) \in P_2(V')$ for each $x \in E'$ and, in fact, the codomain of $\phi'$ must be $P_2(V')$. If $G$ is a graph with loops, the codomain of $\phi'$ must be $P_2(V') \cup P_1(V')$. This definition works equally well if $G$ is a digraph. In that case, the codomain of $\phi'$ must be $V \times V$.

## 1.6 Circuit and Cycle

Let $G = (V, E, \phi)$ be a graph and let $e_1, e_2,...,e_n$ be a trail with vertex sequence $a_1, a_2,...,a_n, a_1$. (returns to the starting point) The subgraph $G'$ of $G$ induced by the set of edges $\{e_1, e_2,...,e_n\}$ is called a circuit of $G$. The length of the circuit is $n$.

- If the only repeated vertices on the trail are $a$, (the start and the end), then the circuit is called a simple circuit or cycle.
- If 'trail' is replaced by directed trail, we obtain a directed circuit and a directed cycle.

**Note:** *In the above definitions, a path is a sequence of edges but a cycle is a subgraph of G. In actual practice, we often think of a cycle as a path, except that it starts and ends at the same vertex. This sloppiness rarely causes trouble, but can lead to problems in formal proofs. Cycles are closely related to the existence of multiple paths between vertices.*

■ **Example 1.7:** *Subgraph-Key Information*

For the graph $G = (V, E, \phi)$ as shown in Fig. 1.13, let $G' = (V', E', \phi')$ be defined by $V' = \{A, B, C\}$, $E' = \{a, b, c, f\}$, and by $\phi'$ being the restriction of $f$ to $E'$ with codomain $P_2(V')$. $\phi'$ is determined completely from knowing $V'$, $E'$ and $\phi$. Thus, to specify a subgraph $G'$, the key information is $V'$ and $E'$.

In the same graph, let $V' = V$ and $E' = \{a, b, c, f\}$. In this case, the vertex $D$ is not a member of any edge of the subgraph. Such a vertex is called an isolated vertex of $G'$.

We may specify a subgraph by giving a set of edges $E' \subseteq E$ and taking $V'$ to be the set of all vertices on some edge of $E'$.

It shows that, $V'$, is the union of the sets $\phi(x)$ over all $x \in E'$. Such a subgraph is called the subgraph induced by the edge set $E'$ or the edge induced subgraph of $E'$. The first subgraph of this example is the subgraph induced by $E' = \{a, b, c, f\}$.

Given a set $V' \subseteq V$, we can take $E'$ to be the set of all edges $x \in E$ such that $\phi(x) \subseteq V'$. The resulting subgraph is called the subgraph induced by $V'$ or the vertex induced subgraph of $V'$. In the given Fig., the edges of the subgraph induced by $V' = \{C, B\}$, are $E' = \{d, e, f\}$

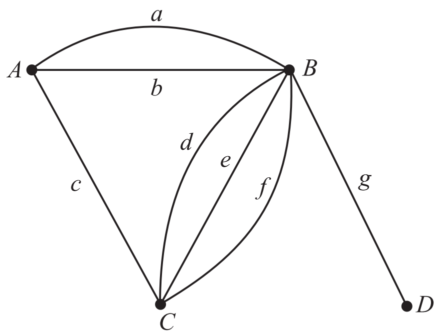



**Fig. 1.13: Parallel Edges**

We consider the path $c$, $a$ with vertex sequence $C, A, B$. The edge $d$ has $\phi(d) = \{C, B\}$. The subgraph $G' = (V', E', \phi'\}$ where $V' = \{C, A, B\}$ and $E' = \{c, a, d\}$ is called a cycle of $G$. Whenever there is a path in $G$, (say $e_1, e_2,...,e_{n-1}$ with vertex sequence $a_1, a_2,...,a_n$) and an edge $x$ with $\phi(x) = \{a_1, a_n\}$, then the subgraph induced by the edges $e_1, e_2,...,e_{n-1}$ $x$ is called the cycle of $G$.

Parallel edges like $a$ and $b$ in the Fig. 1.13 induce a cycle. A loop also induces a cycle. ■

## 1.7 Cycles and Multiple Paths

*Two vertices $u \neq v$ are on a cycle of graph G iff there are at least two paths from u to v that have no vertices in common except the end points u and v.*

To illustrate the above, we consider $u$ and $v$ are on a cycle. We follow the cycle from $u$ to $v$ to obtain one path. Then we follow the cycle from $v$ to $u$ to obtain another. Since the cycle has no repeated vertices, the only vertices that lie on both paths are $u$ and $v$. On the other hand, a path from $u$ to $v$ followed by a path from $v$ to $u$ is a cycle if the paths have no vertices in common other than $u$ and $v$.

## 1.8 Connected Graph

*Let $G = (V, E, \phi)$ be a graph. If for any two distinct elements u and v of V there is a path P from u to v than G is a connected graph. If $|V| = l$, than G is connected.*

To illustrate this definition we refer example 1 which has two distinct pieces. It is not a connected graph. There is, for example, no path from $u = NB$ to $v = AC$. Note that one piece of this graph consists of the vertex induced subgraph of the vertex set $\{AC, SK, BM, RK\}$ and the other piece consists of the vertex induced subgraph of $\{NB, AR, SJ, HB\}$. These pieces are called connected components of the graph. This is the case in general for a graph $G = (V, E, \phi)$.

The vertex set is partitioned into subsets $V_1, V_2,\ldots,V_m$ such that if $u$ and $v$ are in the same subset then there is a path from $u$ to $v$ and if they are in different subsets there is no such path. The subgraphs $G_1 = (V_1, E_1, \phi_1),\ldots,G_m = (V_m, E_m, \phi_m)$ induced by the sets $V_1, V_2,\ldots,V_m$ are called the connected components of $G$. Every edge of $G$ appears in one of the connected components. We can see this by considering that $\{u, v\}$ is an edge and note that the edge is a path from $u$ to $v$ and so $u$ and $v$ are in the same induced subgraph, $G_1$.

■ **Example 1.8:** *Connected Components as an Equivalence Relation*

We have to define firstly, two integers to be 'connected' if they have a common factor. Let 2 and 6 are connected and 3 and 6 are connected, but 2 and 3 are not connected and so we cannot partition the set $V = \{2, 3, 6\}$ into 'connected components'.

We must use some property of definitions of graphs and paths to show that the partitioning of vertices is possible. We can use to do this by constructing an equivalence relation.

For $u, v \in V$, we write $u \sim v$ iff either $u = v$ or there is a walk from $u$ to $v$. It is clear that $\sim$ is reflexive and symmetric. We can prove that it is transitive also. Let $u \sim v \sim w$. The walk from $u$ to $v$ followed by the walk from $v$ to $w$ is a walk from $u$ to $w$. This completes the proof that $u \sim v$ is an equivalent relation. The relation partitions $V$ onto subsets $V_1, V_2,\ldots,V_m$. The vertex induced subgraphs of the $V_i$ satisfy the definition of connected graph.

■

## 1.9 Spanning Subgraph and Induced Subgraph

A spanning subgraph is a subgraph containing all the vertices of $G$.

*i.e.* if $V(H) \subset V(G)$ and $E(H) \subseteq E(G)$ then $H$ is a proper subgraph of $G$ and if $V(H) = V(G)$ then it can be said that $H$ is a spanning subgraph of $G$.

A spanning subgraph need not contain all the edges in $G$.

For any set $S$ of vertices of $G$, the vertex induced subgraph or simply an induced subgraph $< S >$ is the maximal subgraph of $G$ with vertex set $S$. Thus two vertices of $S$ are adjacent in $< S >$ if and only if they are adjacent in $G$.

In other words, if $G$ is a graph with vertex set $V$ and $U$ is a subset of $V$ then the subgraph $G(U)$ of $G$ whose vertex set is $U$ and whose edge set comprises exactly the edge of $E$ which join vertices in $U$ is termed as induced subgraph of $G$.

## 1.10 Eulerian Graph (Eulerian Trail and Circuit)

Let $G = (V, E, \phi)$ be a connected graph. If there is a trail with edge sequence $(e_1, e_2,\ldots, e_k)$ in $G$, which uses each edge in $E$, then $(e_1, e_2,\ldots, e_k)$ is called the Eulerian trail. If there is a circuit $C = (V', E', \phi')$ in $G$ with $E' = E$, then $C$ is called an Eulerian circuit.

We can describe a process for constructing a graph $G = (V, E, \phi)$.

Starting with $V = \{v_1\}$ consisting of a single vertex and with $E = \phi$. Adding an edge $e_{1,}$ with $\phi(e_1) = \{v_1, v_2\}$, to $E$. If $v_1 = v_2$, we have a graph with one vertex and one edge (a loop), else we have a graph with two vertices and one edge. Keeping track of the vertices and edges in the order added. Here $(v_1, v_2)$ is the sequence of vertices in the order added and $(e_1)$ is the sequence of edges in order added. Suppose we continue this process to construct a sequence of vertices (not necessarily distinct) and sequence of distinct edges. At the point where $k$ distinct edges have been added, if $v$ is the last vertex added, then we add a new edge $e_{k+1}$, different from all previous edges, with $\phi(e_{k+1}) = \{v, v'\}$ where either $v'$ is a vertex already added or a new vertex. Here is a picture of this process carried out with the edges numbered in the order added (Fig. 1.14)

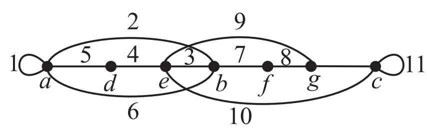



**Fig. 1.14: Eulerian Graph**

where the vertex sequence is

$$S = (a, a, b, e, d, a, b, f, g, e, c, c, g).$$

Such a graph is called a graph with an *Eulerian trail*. The edges, in the order added, are the *Eulerian trail* and $S$ is the vertex sequence of the trail.

By construction, if $G$ is a graph with an eulerian trail, then there is a trail in $G$ that includes every edge in $G$. If there is a circuit in $G$ that includes every edge of $G$ then $G$ is called an Eulerian circuit graph or graph with an Eulerian circuit. Thinking about the above example, if a graph has an Eulerian trail but no Eulerian circuit, then all vertices of the graph have even degree except the start vertex ($a$ in our example with degree 5) and end vertex ($g$ in our example with degree 3). If a graph has an Eulerian circuit then all vertices have even degree. The converses in each case are also true (but take a little work to show): If $G$ is a connected graph in which every vertex has even degree then $G$ has an Eulerian circuit. If $G$ is a connected graph with all vertices but two of even degree, then $G$ has an Eulerian trail joining the two vertices of odd degree.

## 1.11 Hamiltonian Graph

*A cycle in a graph G = (V, E, ϕ) is a Hamiltonian cycle for G in every element of v is a vertex of the cycle. A graph G = (V, E, ϕ) is Hamiltonian if it has a subgraph that is a Hamiltonian cycle for G.*

We can start with a graph $G' = (V, E', \phi')$ that is a cycle and then add additional edges, without adding any new vertices, to obtain a graph $G = (V, E, \phi)$. As an example, consider (Fig. 1.15)

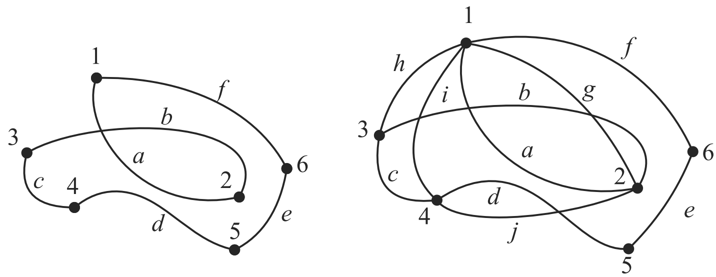



**Fig. 1.15: Hamiltonian Graph**

$(V, E', \phi')$ is the cycle induced by the edges $\{a, b, c, d, e, f\}$. The second graph $G = (V, E, \phi)$ is obtained from $G'$ by adding edges $g$, $h$, $i$ and $j$. A graph that can be constructed from such a two-step process is called a Hamiltonian graph. The cycle $G'$ is called a Hamiltonian cycle of $G$.

## 1.12 Biconnected Graph

Let $G = (V, E, \phi)$ be a graph. For $e, f \in E$ write $e \sim f$ if either $e = f$ or there is a cycle of $G$ that contains both $e$ and $f$. We claim that this is an equivalence relation. The reflexive and symmetric parts are easy. Suppose that $e \sim f \sim g$. If $e = g$, then $e \sim g$, so suppose that $e \neq g$. Let $\phi(e) = \{v_1, v_2\}$. Let $C(e, f)$ be the cycle containing $e$ and $f$ and $C(f, g)$ the cycle containing $f$ and $g$. In $C(e, f)$ there is a path $P_1$ from $v_1$ to $v_2$ that does not contain $e$. Let $x$ and $y \neq x$ be the first and last vertices on $P_1$ that lie on the cycle containing $f$ and $g$. We know that there must be such points because the edge $f$ is on $P_1$. Let $P_2$ be the path in $C(e, f)$ from $y$ to $x$ containing $e$. In $C(f, g)$ there is a path $P_3$ from $x$ to $y$ containing $g$. We claim that $P_2$ followed by $P_3$ defines a cycle containing $e$ and $g$.

Since $\sim$ is an equivalence relation on the edges of $G$, it partitions them. If the partition has only one block, then we say that $G$ is a biconnected graph. If $E'$ is a block in the partition, the subgraph of $G$ induced by $E'$ is called a bicomponent of $G$. Note that the bicomponents of $G$ are not necessarily disjoint: Bicomponents may have vertices in common (but never edges). There are four

bicomponents in the following graph. (Fig. 1.16) Two are the cycles, one is the edge $\{C, O\}$, and the fourth consists of all of the rest of the edges.

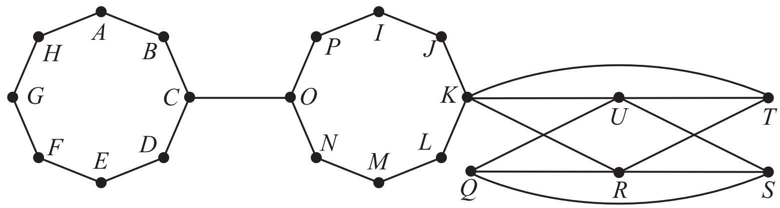



**Fig. 1.16: Biconnected Graph**

## 1.13 Algebraic terms and operations used in Graph Theory

### 1.13.1 *Graphs Homomarphism and Graph Isomorphism*

*Let $G = (V, E, \phi)$ and $G' = (V', E', \phi')$ be two graphs. A homomorphism $f : G \to G'$*

from $G$ to $G'$ is an ordered pair

$f = (f_1, f_2)$ of maps $f_1 : V \to V'$ and $f_2 : E \to E'$ satisfying the following condition:

$\phi(e) = \{u, v\} \Rightarrow \phi'(f_2(e)) = \{f(u), f_1(v)\}$.

i.e. if u and v are the endvertices of $e$ in $G$, then $f_1(u)$ and $f_1(v)$ are the endvertices of $f_2(e)$ in $G'$.

*Let $G_1 = (v_1, E_1)$ and $G_2 = (v_2, E_2)$ be two graphs. A function $f : u_1 \to v_1$ is called a graphs isomorphism iff.*

(*i*) *$f$ is one-to-one and onto.*

(*ii*) *for all $a, b \in v_1$, $\{a, b\} \in E_1$ iff $\{f(a), f(b) \in E_2$ when such a function exists, $G_1$, and $G_2$ are called isomorphic graphs and written as $G_1 \cong G_2$.*

OR

*Two graphs $G_1$ and $G_2$ are called isomorphic to each other if there is a one-to-one correspondence between their vertices and between edges such that incidence relationship is preserve. Written as $G_1 \cong G_2$ or $G_1 = G_2$.*

The necessary conditions for two graphs to be isomorphic are:

(*i*) Both graphs must have the same number of vertices.

(*ii*) Both graphs must have the same number of edges.

(*iii*) Both graphs must have equal number of vertices with the same degree,

(*iv*) They must have the same degree sequence and same cycle vector ($c_1$, $c_2$,...,$c_n$), where $e_i$ is the number of cycles of length $i$.

*e.g.* Fig. 1.17 has isomorphism of graphs.

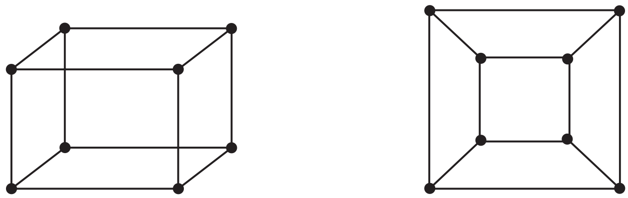
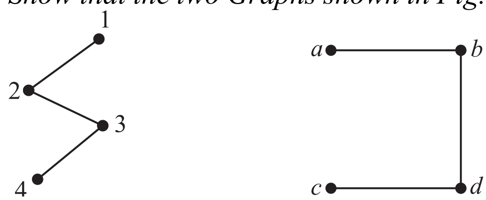


**Fig. 1.17: Isomorphism of Graphs**

■ **Example 1.9:** *Show that the two Graphs shown in Fig. 1.18 are Isomorphic*




($G_1$) ($G_2$)

**Fig. 1.18**

### *Solution:*

Observing the above graphs $G_1$ and $G_2$, we find.

$$V(G_1) = \{1, 2, 3, 4\},\ V(G_2) = \{a, b, c, d\}$$

$$E(G_1) = \{\{1, 2\}, \{2, 3\}, \{3, 4)\}$$

and $$E(G_2) = \{\{a, b\}, \{b, d\}, \{d, c\}\}.$$

Defining a function $f: V(G_1) \rightarrow V(G_2)$ as.

$$f(1) = a, f(2) = b, f(3) = d, \text{ and } f(4) = c.$$

$f$ is clearly **one-one and onto**, hence an isomorphism.

Furthermore, $\{1, 2\} \in E(G_1)$ and $\{f(1), f(2)\} = \{a, a\} \in E(G_2)$

$\{2, 3\} \in E(G_1)$ and $\{f(2), f(3)\} = \{b, d\} \in E(G_2)$

$\{3, 4\} \in E(G_1)$ and $\{f(3), f(4)\} = \{d, c\} \in E(G_2)$

and $\{1, 2\} \notin E(G_1)$ and $\{f(1), f(3)\} = \{a, d\} \notin E(G_2)$

$\{1, 4\} \notin E(G_1)$ and $\{f(1), f(4)\} = \{a, c\} \notin E(G_2)$

$\{2, 4\} \notin E(G_1)$ and $\{f(2), f(4)\} = \{b, c\} \notin E(G_2)$

Hence $f$ preserves adjacency as well as non-adjacency of the vertices.

Therefore, $G_1$ and $G_2$ are isomorphic graphs.

### 1.13.2 Union of two Graphs

If we have two graphs $G_1$ and $G_2$, then, their union will be graph such that

$$V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$$

and $$E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$$

Fig. 1.19 Shows the union operation of two graphs $G_1$ and $G_2$.

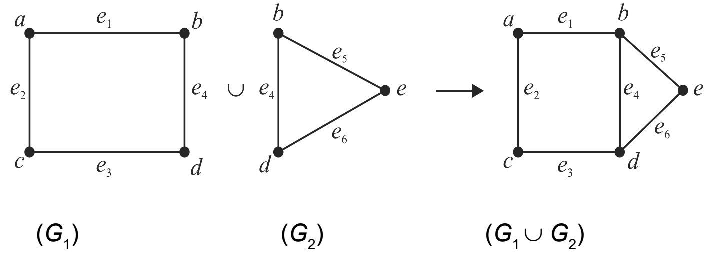



**Fig. 1.19: Union of two Graphs**

### 1.13.3 *Intersection of two Graphs*

If we have two graphs $G_1$ and $G_2$ with at least one vertex is common then their intersection will be a graph such that

$$V(G_1 \cap G_2) = V(G_1) \cap V(G_2)$$

and $$E(G_1 \cap G_2) = E(G_1) \cap E(G_2)$$

Fig. 1.20 illustrates the intersection operation of two graphs $G_1$ and $G_2$.

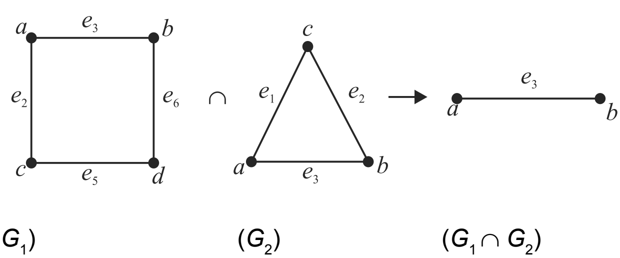



**Fig. 1.20: Intersection of two Graphs**

### 1.13.4 Addition of two Graphs

If we have two graphs $G_1$ and $G_2$ such that $V(G_1) \cap V(G_2) = \phi$, then the sum $G_1 + G_2$ is defined as the graph whose vertex set is $V(G_1) + V(G_2)$ and the edge set is consisting these edges, which are $G_1$ and in $G_2$ and the edges contained, by joining each vertex of $G_1$ to each vertex of $G_2$. Fig. 1.21 illustrates the addition of two graphs $G_1$ and $G_2$.

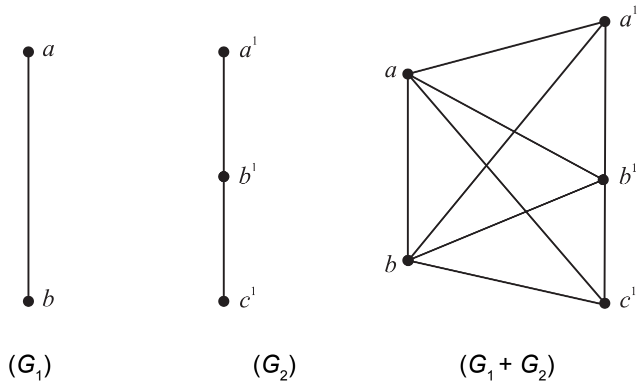



**Fig. 1.21: Addition of two Graphs**

### 1.13.5 Direct Sum or Ring Sum of two Graphs

If we have two graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$. Then the direct sum of $G_1$ and $G_2$ (denoted by $G_1$, $G_2$) is defined as the graph $G$ such that

(*i*) $V(G) = V(G_1) \cup V(G_2)$

(*ii*) $E(G) = E(G_1) \cup E(G_2) - E(G_1) \cap E(G_2)$

*i.e.* the edges that either in $G_1$ or $G_2$ but not in both. The direct sum is illustrated in Fig. 1.22 for two graphs $G_1$ and $G_2$.

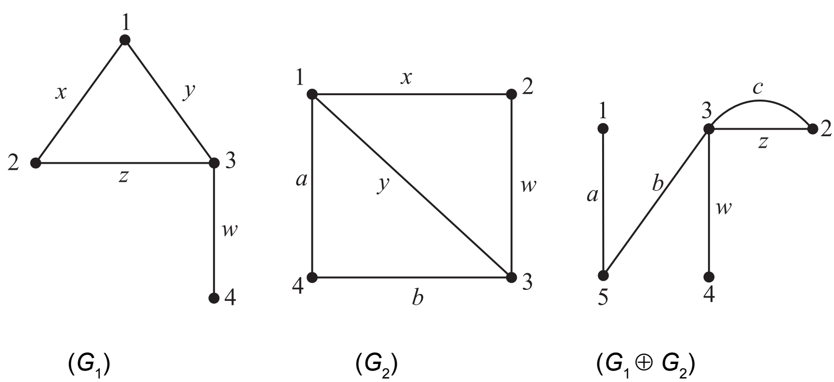



**Fig. 1.22: Ring Sum of two Graphs**

### 1.13.6 *Product of two Graphs*

We can define the product of two graphs $G_1 \times G_2$ by considering any two points $u = (u_1, u_2)$ and $v = (v_1, v_2)$ is $V = V_1 \times V_2$. Then $u$ and $v$ are adjacent in $G_1 \times G_2$ whenever $[u_1 = v_1$ and $u_2$ adj. $v_2]$ or $[v_1$ and $u_1$ adj $v_1]$.

Fig. 1.23 illustrates the product of two graphs ($G_1 \times G_2$).

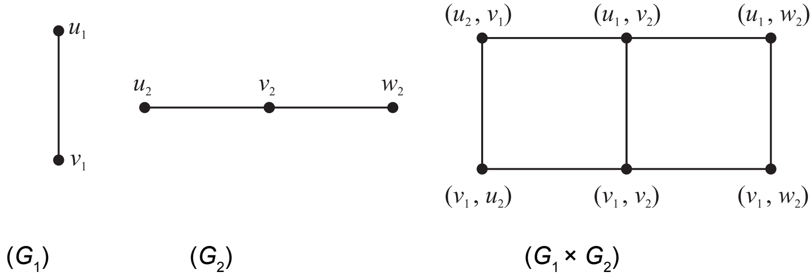



**Fig. 1.23: Products of two Graphs**

### 1.13.7 *Composition of two Graphs*

The composition $G = G_1[G_2]$ also has $v = v_1 \times v_2$ as its point set, and $u = (u_1, u_2)$ is adjacent with $v = (v_1, v_2)$ whenever ($u_1$ adj. $v_1$) or ($u_1 = v_1$ and $u_2$ adj. $v_1$).

Fig. 1.24 illustrates the composition of two graphs.

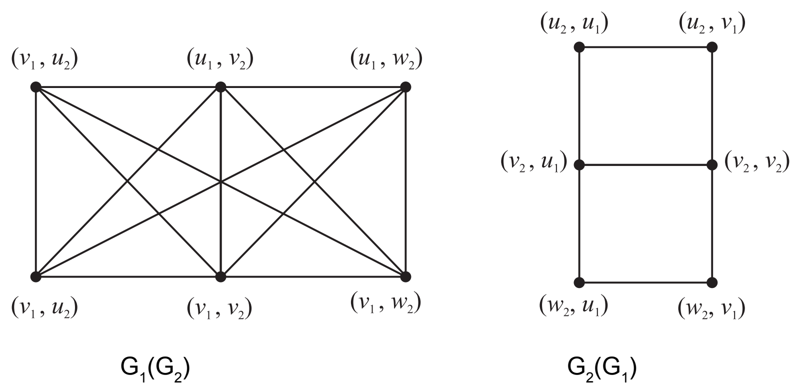



**Fig. 1.24: Composition of two Graphs**

### 1.13.8 *Complement of a Graph*

The complement $G'$ of a graph $G$ may be defined as a simple graph with the same vertex set as $G$ and where two vertices $u$ and $v$ adjacent only when they are not adjacent in $G$.

Fig. 1.25(a) shows the complement of a graph

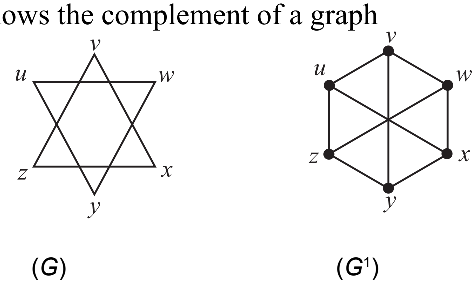



**Fig. 1.25 (a): Complement of a Graph**

A graph $G$ is *self-complementary* if it is isomorphic to its complement. Fig. 1.25(b) shows this situation.

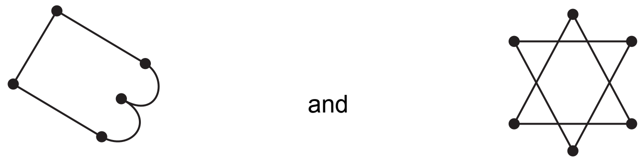



**Fig. 1.25 (b): Self Complement of a Graph**

The self-complementary graph with five vertices is (as shown in Fig. 1.25(c))

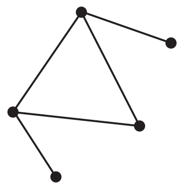


**Fig. 1.25 (c): Self Complementary with Five Vertices**

## 1.13.9 Fusion of a Graph

A pair of vertices $v_1$ and $v_2$ in a graph $G$ is said to be '*fused*' if these two vertices are replaced by a single new vertex $v$ s.t. every edge that was adjacent to either $v_1$ or $v_2$ on both is adjacent $v$.

We can observe that the fusion of two vertices does not alter the number of edges of graphs but reduced the vertices by one. Fig. 1.26 shows this kind of fusion.

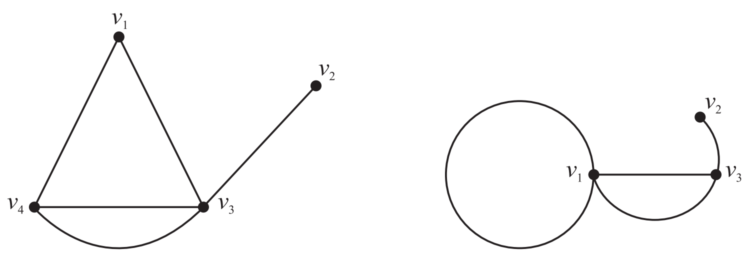



**Fig. 1.26: Fusion of Graph**

## 1.13.10 Rank and Nullity

Let $G$ be a graph with n vertices, $m$ edges and $k$ components. We define the rank $P(G)$ and nullity $\mu(G)$ of $G$ as follows:

$$P(G) = \text{Rank of } G = n - k$$

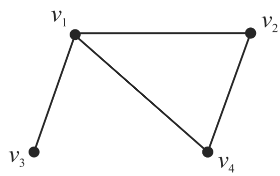
$$\mu(G) = \text{Nullity of } G = m - n + k$$

If $G$ is connected Hence,

$$P(G) = n - 1 \text{ and } \mu(G) = m - n + 1.$$

### 1.13.11 *Adjacency Matrix*

A graph can be represented by an adjacency matrix. In particular, if a graph has vertices $v_1,...,v_n$, then the adjacency matrix is $n \times n$. The entry in row $i$, column $j$ is 1 if there is an edge $v_i–v_j$ and is 0 if there is no such edge. For example, here is a graph and its adjacency matrix:




$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

The adjacency matrix of an undirected graph is always symmetric about the diagonal line running from the upper left entry to the lower right. The adjacency matrix of a directed graph need not be symmetric, however. Entries on the diagonal of an adjacency matrix are nonzero only if the graph contains self-loops.

Adjacency matrices are useful for two reasons. First, they provide one way to represent a graph in computer memory. Second, by mapping graphs to the world of matrices, one can bring all the machinery of linear algebra to bear on the study of graphs. For example, one can analyze a highly-prized quality of graphs called "expansion" by looking at eigenvalues of the adjacency matrix. (In a graph with good expansion, the number of edges departing each subset of vertices is at least proportional to the size of the subset. This is not so easy to achieve when the graph as a whole as few edges, say $|E| = 3|V|$.) Here we prove a simpler theorem in this vein. If $M$ is a matrix, then $M_{ij}$ denotes the entry in row $i$, column $j$. Let $M^k$ denote the $k$-th power of $M$. As a special case, $M^0$ is the identity matrix.

### 1.13.12 Some Important Theorems

#### *Theorem 1.4*

The maximum number of edges in a simple graph with $n$ vertices is $\dfrac{n(n-1)}{2}$

#### *Proof:*

By handshaking theorem, we have

$$\sum_{i=1}^{n} f(v_i) = 2e$$

Where $e$ is the number of edges with vertices in the graph $G$.

$$d(v_1) + d(v_2) + ,\ldots, + d(v_n) = 2e. \qquad ...(i)$$

Since we know that maximum degree of each vertex in a graph $G$ can be $(n - 1)$.

Therefore, equ (i) reduces to

$(n-1)+(n-1)+,\ldots,$ to $n$ terms $= 2e$

$$n(n-1) = 2e$$

$$e = \frac{n(n-1)}{2}$$

Hence the maximum number of edges in any simple graph with $n$ vertices is $\frac{n(n-1)}{2}$. ■

### Theorem 1.5

*The maximum number of lines among all p point graphs with no triangles is* $\left[\frac{p^2}{4}\right]$

### Proof:

Suppose the statement is true for all even $p \leq 2n$.

We then prove for $p = 2n + 2$.

Let $G$ be a graph with $p = 2n + 2$ points and no triangles.

Since $G$ is not totally disconnected, there are adjacent points $u$ and $v$.

The subgraph $G' = G - \{u, v\}$ has $2^n$ points and no triangles, so that by the induction hypothesis.

$G'$ has at most $\left[\frac{4n^2}{4}\right] = n^2$ lines.

There might not be a point $w$ *s.t.,* $u$ and $v$ are both adjacent to $w$, for then $u$, $v$ and $w$ would be points of a triangle in $G$.

If $u$ is adjacent to $k$ points of $G'$, $v$ can be adjacent to at most $2n - k$ points.

Thus $G$ has at most

$n^2 + k + (2n - k) + 1 = n^2 + 2n + 1 = \frac{p^2}{4} = \left[\frac{p^2}{4}\right]$ lines. ■

### Theorem 1.6

*A simple graph with n vertices must be connected if it has more than* $\frac{(n-1)(n-2)}{2}$ *edges.*

### *Proof:*

Let $G$ be a simple graph on $n$ vertices.

We can choose $n-1$ vertices $v_1, v_2, \ldots, v_{n-1}$ of $G$.

We have maximum $\binom{n-1}{2} = \frac{(n-1)(n-2)}{2}$ number of edges.

Only can be drawn between there vertices.

If we have more than $\frac{(n-1)(n-2)}{2}$ edges, at least one edge should be drawn between the $n^{\text{th}}$ vertex $v_n$ to some vertex $v_i$, $1 \le i \le n-1$ of $G$.

Hence $G$ must be connected.

### *Theorem 1.7*

*A simple graph with n vertices and k components cannot have more than* $\frac{(n-k)(n-k+1)}{2}$ *edges. If* $m > \frac{1}{2}(n-1)(n-2)$ *then a simple graph with n vertices and m edges in connected.*

### *Proof:*

Let $n_i$ the number of vertices is component $i$, $i \le i \le k$

Then $$\sum_{i=1}^{k} n_i = n$$

A component with $n_i$ vertices will have the maximum possible number of edges when it is complete.

*i.e.*, it contains $\frac{1}{2} n_i (n_i - 1)$ edges.

The maximum number of edges is

$$\frac{1}{2}\sum_{i=1}^{k} n_i (n_i - 1) = \frac{1}{2}\sum_{i=1}^{k} n_i^2 - \frac{1}{2}\sum_{i=1}^{k} n_i$$

$$\le \frac{1}{2}\left[n^2 - (k-1)(2n-k)\right] - \frac{1}{2}n$$

$$= \frac{1}{2}\left[n^2 - 2nk + k^2 + n - k\right]$$

$$= \frac{1}{2}(n-k)(n-k+1)$$

Suppose graph is disconnected. Then it has at least two components,

Therefore, $m \le \frac{1}{2}(n-k)(n-k+1)$ for $k \ge 2$.

$$\le \frac{1}{2}(n-2)(n-1)$$

This contradiction assumes that

$$m = \frac{1}{2}(n-1)(n-2)$$

Therefore, the graph is connected. ■

### *Theorem 1.8*

*Every graph G = (V, E) has at least |V| – |E| connected components.*

### *Proof:*

We use induction on the number of edges. Let $P(n)$ be the proposition that every graph $G = (V, E)$ with $|E| = n$ has at least $|V| - n$ connected components.

*Base case:* In a graph with 0 edges, each vertex is itself a connected component, and so there are exactly $|V| - 0 = |V|$ connected components.

*Inductive step:* Now we assume that the induction hypothesis holds for every $n$-edge graph in order to prove that it holds for every $(n + 1)$ — edge graph, where $n \ge 0$. Consider a graph $G$ $(V, E)$ with $n + 1$ edges. Remove an arbitrary edge $u$-$v$ and call the resulting graph $G'$. By the induction assumption, $G'$ has at least $|V| - n$ connected components. Now add back the edge $u - v$ to obtain the original graph $G$. If $u$ and $v$ were in the same connected component of $G'$, then $G$ has the same number of connected components as $G'$, which is at least $|V| - n$. Otherwise, if $u$ and $v$ were in different connected components of $G'$, which is at least $|V| - n$. Otherwise, if $u$ and $v$ were in different connected components of $G'$, then these two components are merged into one in $G$, but all other components remain. Therefore, $G$ has at least $|V| - n - 1 = |V| - (n + 1)$ connected components. ■

***Corollary:*** *Every connected graph with n vertices has at least n – 1 edges.*

A couple points about the proof of Theorem 7 are worth noting. First, notice that we used induction on the number of edges in the graph. This is very common in proofs involving graphs, and so is induction n the number of vertices.

The second point is more subtle. Notice that in the inductive step, we took an arbitrary $(n + 1)$-edge graph, threw out an edge so that we could apply the induction assumption, and then put the edge back. This might seem like needless effort; why not start with an $n$-edge graph and add one more to get an $(n + 1)$-

edge graph? That would work fine in this case, but opens the door to a very nasty logical error in similar arguments. ■

### *Theorem 1.9*

*A connected graph has an Euler tour if and only if every vertex has even degree.*

### *Proof:*

If a graph has an Euler tour, then every vertex must have even degree; in particular, a vertex visited $k$ times on an Euler tour must have degree $2k$.

Now suppose every vertex in graph $G$ has even degree. Let $W$ be the longest walk in $G$ that traverses every edge at most once:

$$W = v_0, v_0 - v_1, v_1, v_1 - v_2, v_2, \ldots, v_{n-1} - v_n, v_n$$

The walk $W$ must traverse every edge incident to $v_n$; otherwise, the walk could be extended. In particular, the walk traverses two of these edges each time it passes through $v_n$ and traverses $v_{n-1} - v_n$ at the end of the walk. This accounts for an odd number of edges, but the degree of $v_n$ is even by assumption. Therefore, the walk must also begin at $v_n$; that is, $v_0 = v_n$.

Suppose that $W$ is not an Euler tour. Because $G$ is a connected graph, we can find an edge not in $W$ but incident to some vertex in $W$. Call this edge $u - v_i$. But then we can construct a longer walk:

$$u, u - v_i, v_i, v_i - v_i - v_{i+1}, \ldots, v_{n-1} - v_n, v_n, v_0 - v_1, \ldots, v_{i-1} - v_i, v_i$$

This contradicts the definition of $W$, so $W$ must be an Euler tour after all.

***Corollary:*** *A connected graph has an Euler walk if and only if either 0 or 2 vertices have odd degree.*

Hamiltonian cycles are the unruly cousins of Euler tours. A Hamiltonian cycle is walk that starts and ends at the same vertex and visits every vertex in a graph exactly once. There is no simple characterization of all graphs with a Hamiltonian cycle. (In fact, determining whether a given graph has a Hamiltonian cycle is "NP-complete".) ■

### *Theorem 1.10*

*Let $G$ be a digraph (possibly with self-loops) with vertices $v_1,\ldots,v_n$. Let $M$ be the adjacency matrix of $G$. Then $M^k_{ij}$ is equal to the number of length-$k$ walks from $v_i$ to $v_j$.*

### *Proof:*

We use induction on $k$. The induction hypothesis is that $M^k_{ij}$ is equal to the number of length-$k$ walks from $v_i$ to $v_j$, for all $i, j$.

Each vertex has a length-0 walk only to itself. Since $M_{ij}^{k} = 1$ if and only if $i = j$, the hypothesis holds for $k = 0$.

Now suppose that the hypothesis holds for some $k \geq 0$. We prove that it also holds for $k + 1$. Every length-($k + 1$) walk from $v_i$ to $v_j$ consists of a length $k$ walk from $v_i$ to some intermediate vertex $v_m$ followed by an edge $v_m$–$v_j$. Thus, the number of length-($k + 1$) walks from $v_i$ to $v_j$ is equal to:

$$M_{iv1}^{k} M_{v1j} + M_{iv2}^{k} M_{v2j} + \ldots + M_{ivn}^{k} M_{vnj}$$

This is precisely the value of $M_{ij}^{k+1}$, so the hypothesis holds for $k + 1$ as well. The theorem follows by induction.

## 1.14 Some Popular Problems in Graph Theory

### 1.14.1 *Tournament Ranking Problem*

Suppose that $n$ players compete in a round-robin tournament. Thus, for every pair of players $u$ and $v$, either $u$ beats $v$ or else $v$ beats $u$. Interpreting the results of a round-robin tournament can be problematic. There might be all sorts of cycles where $x$ beat $y$, $y$ beat $z$, yet $z$ beat $x$. Graph theory provides at least a partial solution to this problem.

The results of a round-robin tournament can be presented with a tournament graph. This is a directed graph in which the vertices represent players and the edge indicate the outcomes of games. In particular, an edge from $u$ to $v$ indicates that player $u$ defeated player $v$. In a round-robin tournament, every pair of players has a match. Thus, in a tournament graph there is either can edge from $u$ to $v$ or an edge from $v$ to $u$ for every pair of vertices $u$ and $v$. Here is an example of a tournament graph (refer Fig. 1.27)

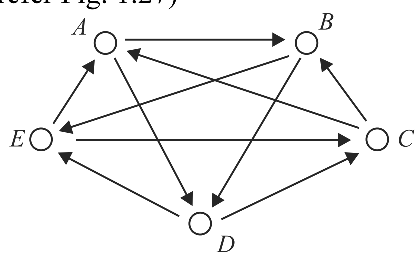



**Fig. 1.27: Tournament graph**

The notions of walks, Euler tours, and Hamiltonian cycles all carry over naturally to directed graphs. A directed walk is an alternating sequence of vertices and directed edges:

$v_0, v_0 \rightarrow v_1, v_1, v_1 \rightarrow v_2, v_2, \ldots, v_{n-1}, v_{n-1} \rightarrow v_n, v_n$

A directed Hamiltonian path is a directed walk that visits every vertex exactly once.

We're going to prove that in every round-robin tournament, there exists a ranking of the players such that each player lost to the player ranked one position higher. For example, in the tournament above, the ranking.

$$A > B > D > E > C$$

satisfies this criterion, because $B$ lost to $A$, $D$ lost to $B$, $E$ lost to $D$ and $C$ lost to $E$. In graph terms, proving the existence of such a ranking amounts to proving that every tournament graph has a Hamiltonian path.

### *Theorem 1.11*

*Every tournament graph contains a directed Hamiltonian path.*

### *Proof:*

We use strong induction. Let $P(n)$ be the proposition that every tournament graph with n vertices contains a directed Hamiltonian path.

*Base case:* $P(1)$ is trivially true; every graph with a single vertex has a Hamiltonian path consisting of only that vertex.

*Inductive step:* For $n \geq 1$, we assume that $P(1),..., P(n)$ are all true and prove $P(n + 1)$. Consider a tournament with $n + 1$ players. Select one vertex $v$ arbitrarily. Every other vertex in the tournament either has an edge to vertex $v$ or an edge from vertex $v$. Thus, we can partition the remaining vertices into two corresponding sets, $T$ and $F$, each containing at most $n$ vertices. (Fig. 1.28)

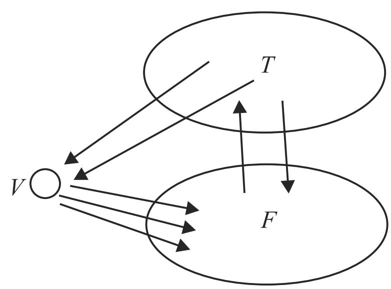



**Fig. 1.28: Tournament Graph as Hamiltonian Path**

The vertices in $T$ together with the edges that join them form a smaller tournament. Thus, by strong induction, there is a Hamiltonian path within $T$. Similarly, there is a Hamiltonian path within the tournament on the vertices in $F$. Joining the path in $T$ to the vertex $v$ followed by the path in $F$ gives a Hamiltonian path through the whole tournament. (As special cases, if $T$ or $F$ is empty, then so is the corresponding portion of the path.) ■

***Note:*** *The ranking defined by a Hamiltonian path is not entirely satisfactory. In the example tournament, notice that the lowest-ranked player (C) actually defeated the highest-ranked player (A).*

### 1.14.2 *The Königsberg Bridge Problem*

The Königsberg bridge problem is perhaps the best-known example in Graph Theory. It was a long-standing problem until solved by Leonhard Euler (1707-1783) in 1736, by means of a graph. Euler wrote the first paper even in graph theory and thus became the origination of Graph Theory as well as of the rest of topology. The city of Königsberg was located on the Pregel river in Prussia. The city occupied the island of kneiphopf plus areas on both banks. These regions were linked by seven bridges as shown in Fig 1.29. The citizens wondered whether they could leave home, cross every bridge exactly once, and return home. The problem reduces to traversing the Fig. (right) with heavy dots representing land masses and cures representing bridges.

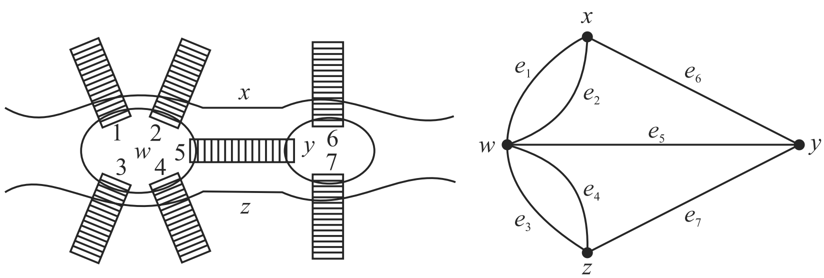



**Fig. 1.29: The Königsberg Bridge Problem**

Two stands $w$ and $y$, formed by the Pregel river in Königsberg (Then the capital of east Prussia but now renamed Kalinongrad and in West Soviet Russia) were connected to each other to the banks $x$ and $z$ with seven budges as shown in Fig. The problem was to start any of the four land areas of the city $w$, $x$, $y$ or $z$, walk over each of the seven bridges exactly once, and return to the starting point (without swimming across the river). Euler represented this situation by means of a graph as shown in Fig.. The vertices represent the land areas and the edges represent the bridges.

Later on the problem was solved by introducing a new bridge on left most side.

### 1.14.3 Four Colour Problem

One of the most famous problem of Graph Theory is the four colour problem. This problem states that any map on a plane on the surface of a sphere can be coloured with four colours in such a way that no two adjacent countries or states have the same colour. This problem can be translated as a problem in Graph Theory. We represent each country or state by a point and join two points by a line if the countries are adjacent. The problem is to colour the points in each way that adjacent points have different colours. This problem was first posed in 1852 by Frances Gutherie a post-graduate student at the University College,

London. This problem was finally proved by Appel and Hahen in 1976 and they have used 400 pages of arguments and about 1200 hours of computer time on some of the best computers in the world to arrive the solution.

### 1.14.4 Three Utilities Problem

We consider three houses $H_1$, $H_2$ and $H_3$, each to be connected to each of the three utilities viz. water ($W$), gas ($G$), and electricity ($E$) by means of conducts. It is possible to make such connections without any crossovers of the conduits. We can see in Fig. 1.30. How this problem can represented by a graph the conduits are shown as edges while the houses and utilities supply centres are vertices. The graph (right) cannot be drawn n the plane without edge crossing over. The answer of this problem is no.

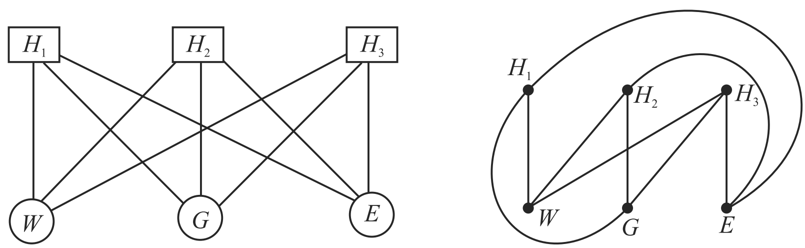



**Fig. 1.30: Three Utilities Problem**

### 1.14.5 Traveling - Salesman Problem

A salesman requires to visit a number of cities during his assignment. Given the distance between the cities, in what order should be travel as to visit every city precisely once and return home, with the minimum mileage travelled?

We represent the cities by vertices and the road between them by edges. We get a graph.

In this graph, with every edge $e_i$ there is associated a real number $W(e_i)$ being the weight of edge $e_i$.

### *Illustration:*

Suppose that a salesman wants to visit five cities, namely $A$, $B$, $C$, $D$ and $E$ (as shown in Fig. 1.31). In which order should he visit these cities to travel the minimum total distance? To solve this problem we may assume the salesman starts in $A$ (since this must be a part of circuit) and examine all possible ways for him to visit the other four cities and then return to $A$. There exist a total of 24 such circuits, but since we travel the same distance when we travel a circuit in reverse order, we need to consider only 12 different circuits to find the minimum total distance he must travel. We can list these 12 different circuits and the total distance travelled for each circuit. The routes can be found the distance travelled. The travelling salesman problem asks for a circuit of minimum total weight

in a weighted, complete, undirected graph that visits each vertex exactly once and returns to its starting point. This is equivalent to asking for a Hamiltonian circuit with minimum total weight in the complete graph, since each vertex is visited exactly once in the circuit.

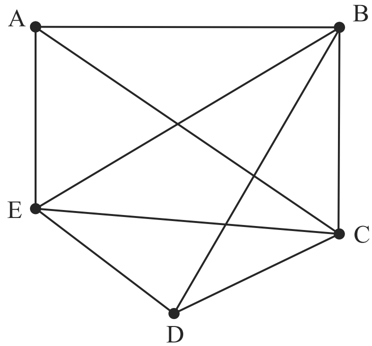



**Fig. 1.31: Traveling Salesman Problem**

The most straight way to solve an instance of this problem is to examine all possible Hamiltonion circuits and select one of minimum total length. (Fig. 1.31)

If we have $n$ vertices in the graph. Once a starting point is chosen, there are $(n–1)!$ different Hamiltonion circuits to be examined, since there are $(n–1)$ choices for second vertex, $(n – 2)$ choices for the third vertex and, so on.

A Hamiltonion circuit can be travelled in reverse order, we can only examine

$\frac{(n-1)!}{2}$ circuits to reach the answer. $\frac{(n-1)!}{2}$ grown extremely rapidly.

### 1.14.6 *MTNL'S Networking Problem*

Suppose Mahanagar Telephone Nigam Limited (MTNL) is interested in identifying those lines that must stay in services to avoid disconnecting the network as shown in Fig. 1.32. Which is a typical problem of graph theory.

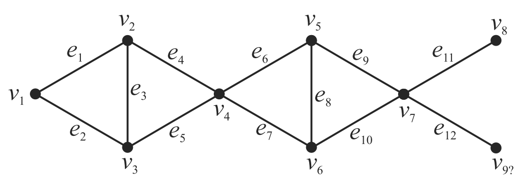



**Fig. 1.32: MTNL Network Problem**

### 1.14.7 *Electrical Network Problems*

The properties of an electrical network are functions of only two key factors:

(*i*) The nature and value of the elements forming the network, such as resistors, inductors and transistors etc.

(*ii*) The way these elements are connected together *i.e.* the topology of the network.

Since these are a few different types of elements of electrical circuit, the variations in networks are due to variations in networks are due to variation in technology.

The particular topology of an electrical network is studied by means of graph theory. While drawing a graph of an electrical network the junctions are represented by vertices, and the branches are represented by edges, regardless the nature and size of the electrical elements shown in Fig. 1.33.

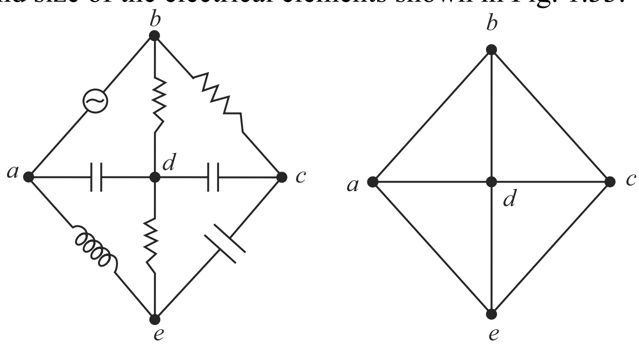



**Fig. 1.33: Electrical Network Problem**

## 1.14.8 *Satellite Channel Problem*

Suppose six TV companies apply for frequency allotment. If the relay centre of these companies are not less than 1000 km then only the same frequency could be alloted without interference. The aim is to assign as small number of different frequency as possible. The Fig. 1.34 shows the line joining two companies to show that they came relay centres less than 1000 km apart. This is a typical problem of graph theory. As an obvious case, we could assign same frequency to company 1 and 4, 2 and 5, and 3 and 6 respectively.

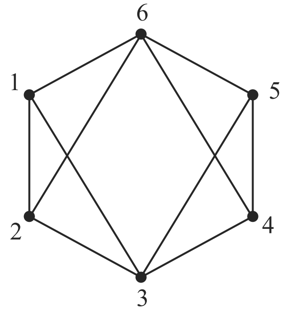



**Fig. 1.34: Relay Centres and their Distance**

## 1.15 Applications of Graphs

Graphs are the most useful mathematical objects in computer science. We can model an enormous number of real-world systems and phenomena using graphs. Once we have created such a model, we can tap the vast store of theorems about graphs to gain insight into the system we are modeling. Here are some practical situations where graphs arise:

*Data Structures:* Each vertex represents a data object. There is a directed edge from one object to another if the first contains a pointer or reference to the second.

*Attraction:* Each vertex represents a person, and each edge represents a romantic attraction. The graph could be directed to model the unfortunate asymmetries.

*Airline Connections:* Each vertex represents an airport. If there is a direct flight between two airports, then there is an edge between the corresponding vertices. These graphs often appear in airline magazines.

*The Web:* Each vertex represents a web page. Directed edges between vertices represent hyperlinks.

People often put numbers on the edges of graph, put colors on the vertices, or add other ornaments that capture additional aspects of the phenomenon being modeled. For example, a graph of airline connections might have numbers on the edges to indicate the duration of the corresponding flight. The vertices in the attraction graph might be colored to indicate the person's gender.

## 1.16 Worked Examples

■ **Example 1.10:** *Determine the number of edges in a graph with 6 vertices, 2 of degree 4 and 4 of degree 2. Plot two such graphs.* (Delhi, (MCA), 2004)

Solution we consider a graph with 6 vertices and having e number of edges from Handshaking lemm, we have.

$\Rightarrow d(v_1) + d(v_2) + d(v_3) + d(v_4) + d(v_5) + d(v_6) = 2e.$

given, 2 vertices are of degree 4 and 4 vertices are of degree 2.

Hence the above equation

$(4 + 4) + (2 + 2 + 2 + 2) = 2e.$

$\Rightarrow 16 = 2e \Rightarrow e = 8.$

Hence the number of edges in a graph with 6 vertices with the given conditions is 8.

*E.g.*

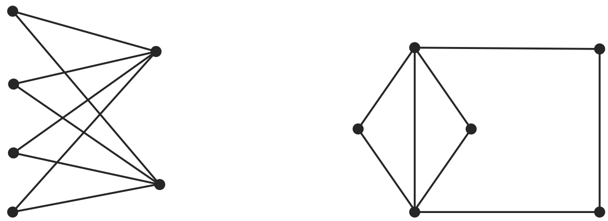
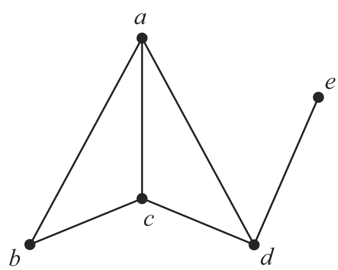
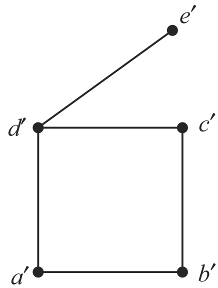


**Fig. 1.35**

■

■ **Example 1.11:** *Show that the following graphs are isomorphic*




### *Solution:*

We define the function $f : G \to G'$ between two graphs.

The following are the degree of $G$ and $G'$

| | |
|---|---|
| deg. $(G)$ | deg $(G')$ |
| $\deg(a) = 3$ | $\deg(a') = 3$ |
| $\deg(b) = 2$ | $\deg(b') = 2$ |
| $\deg(c) = 3$ | $\deg(c') = 3$ |
| $\deg(d) = 3$ | $\deg(d') = 3$ |
| $\deg(e) = 1$ | $\deg(e') = 1$ |

Each graph has 5-vertices and 6-edges,

$$d(a) = d(a') = 3$$
$$d(b) = d(b') = 2$$
$$d(c) = d(c') = 3$$
$$d(d) = d(d') = 3$$
$$d(e) = d(e') = 1$$

Hence the correspondence is $a - a'$, $b - b'$,...,$e - e'$

which is one-to-one

Hence the given graphs are isomorphic. ■

■ **Example 1.12:** *Find the rank and nullity of the complete graph $k_n$.*

***Solution:***

Since $k_n$ is a connected graph with n vertices

$$m = \frac{n(n-1)}{2} \text{ edges.}$$

Therefore, by the definition of rank and nullity, we have

Rank of $k_n = n - 1$

$$\text{Nullity of } k_n = m - n + 1 = \frac{1}{2}n(n-1) - n + 1$$

$$= \frac{1}{2}(n-1)(n-2)$$ ■

■ **Example 1.13:** *If G be a simple graph with n vertices and m edges where m is at least 3.*

*If* $m \geq \frac{1}{2}(n-1)(n-2) + 2.$

*Prove that G is Hamiltonian. Is the converse true?*

***Solution:***

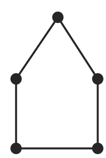
Let $u$ and $v$ be any two non-adjacent vertices in $G$, and $x$, $y$ be their respective degrees.

If we delate $u$, $v$ from $G$, we get a subgraph with $(n-2)$ vertices.

If this subgraph has $q$ edges, then $q \leq \frac{1}{2}(n-2)(n-3)$

Since $u$ and $v$ are non-adjacent, then

$$m = q + x + y.$$



$$\text{Thus, } x + y = m - q \geq \left\{\frac{1}{2}(n-1)(n-2) + 2\right\} - \left\{\frac{1}{2}(n-2)(n-3)\right\}$$

$$= n$$

Hence the graph is Hamiltonion.

The converse of the result may or may not true because, a 2-regular graph with 5-vertices is Hamiltonian but the inequality does not hold. ■

■ **Example 1.14:** *Show that the following sequence is graphical. Also find a graph corresponding to the sequence,* 6, 5, 5, 4, 3, 3, 2, 2, 2.

***Solution:***

The given sequence can be reduced as under:

Given sequence 6, 5, 5, 4, 3, 3, 2, 2, 2

On reducing first six terms by 1 counting from second term, we get

4, 4, 3, 2, 2, 1, 2, 2.

Which can be written in decreasing order as,

4, 4, 3, 2, 2, 2, 2, 1.

On reducing first three terms by 1, counting from second, we get

3, 2, 1, 1, 2, 2, 1

Which can be written in decreasing order as,

3, 2, 2, 2, 1, 1, 1

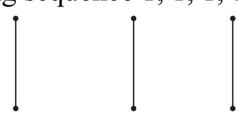
On reducing first three terms by 1, counting from second, we get

1, 1, 1, 1, 1, 1

Now sequence 1, 1, 1, 1, 1, 1 is graphical

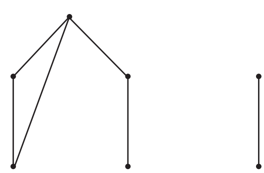
Hence the given sequence can be concluded as a graphical sequence.

The graph of corresponding sequence 1, 1, 1, 1, 1, 1 is given below:



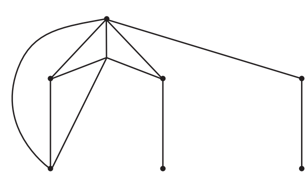
We can obtain a graph corresponding to the given sequence, adding a vertex to each of the vertices of the vertices whose degrees are $t_1$–1, $t_2$–1, ...... , $t_s$–1 and the process can be repeated.

**Step 1:**



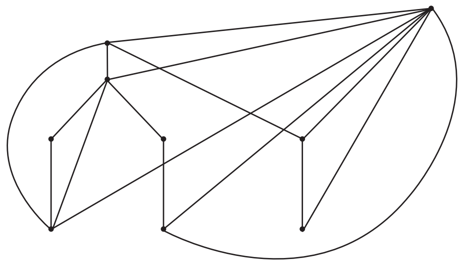
Degree sequences of this graph is 3, 2, 2, 2, 1, 1, 1.

**Step 2:**



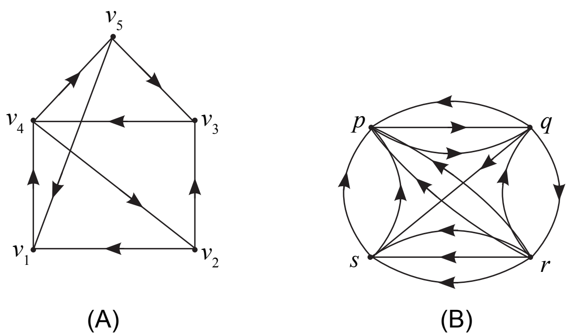
Degree sequences of this graph is 4, 4, 3, 2, 2, 2, 2, 1.

**Step 3:** Final graph can be plotted as;



Degree sequences of this graph is 4, 4, 3, 2, 2, 2, 2, 1. ■

■ **Example 1.15:** *Find in-degree and out-degree of each vertex of the following directed graph.*




***Solution:***

(A)

| Vertex | in-degree | Out-degree |
|---|---|---|
| $v_1$ | 2 | 1 |
| $v_2$ | 1 | 2 |
| $v_3$ | 2 | 1 |
| $v_4$ | 2 | 2 |
| $v_5$ | 1 | 2 |

(B)

| Vertex | in-degree | Out-degree |
|---|---|---|
| $p$ | 5 | 2 |
| $q$ | 3 | 3 |
| $r$ | 1 | 6 |
| $s$ | 4 | 2 |

■

■ **Example 1.16:** *Find the total number of subgraphs and spanning subgraphs in $K_6$, $L_5$ and $Q_3$.*

### *Solution:*

In a graph $K_6$

We have $|V| = 6$ and $|E| = 15$

then, total number of subgraph is $(2^{|V|} - 1) \times 2^{|E|}$

$$(2^6 - 1) \times 2^{15} = 63 \times 32768 = 2064384$$

(The total number of spanning subgraphs is $2^{15} = 32768$.

In a **Linear Graph** ($L_5$).

We have $|V| = 5$ and $|E| = 4$

Then, total number of subgraph is $(2^{|V|} - 1)\ 2^{|E|}$

$$= (2^5 - 1) \times 2^4 = 31 \times 16 = 496$$

The total number of spanning subgraph is $2^4 = 16$

In a **3-Cube graph** ($Q_3$),

we have, $|V| = 8$ and $|E| = 12$

Then, total number of subgraph is $(2^{|V|} - 1) \times 2^{|E|}$

$$= (2^8 - 1) \times 2^{12} = 127 \times 4096 = 520192.$$

(Total number of spanning subgraph is $2^{12} = 4096$) ■

■ **Example 1.17:** *Draw the undirected graph a corresponding to adjacency matrix.*

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

### *Solution:*

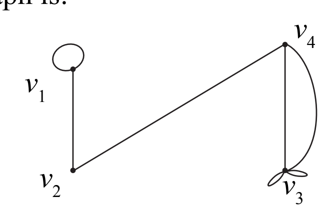
The given adjacency matrix is a square matrix of order 4.

Let G has four vertices $v_1$, $v_2$, $v_3$ and $v_4$.

Since can draw n edges from $v_i$ to $v_j$ where $a_{ij} = n$

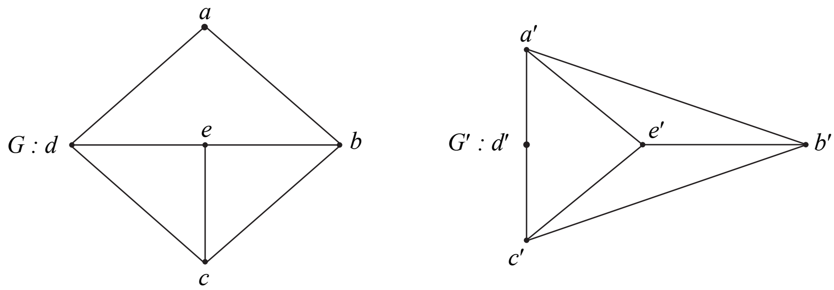
Also we can draw $n$ loop at $v_i$ where $a_{ij} = n$,

Hence the required graph is:




■

■ **Example 1.18:** *Show that the graph G and G′ are isomorphic.*




***Solution:***

We con consider the map $f: G \to G'$ and define as $f(a) = d'$, $f(b) = a'$, $f(c) = b'$, $f(d) = c'$ and $f(e) = e'$

The adjacency matrix of $G$ for the ordering $a, b, c, d$ and $e$ can be written as;

$$A(G) = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \!\!\begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

The adjacency matrix of $G'$ for the ordering $d'$, $a'$, $b'$, $c'$ and $e'$ may be written as.

$$A(G') = \begin{array}{c} \\ d' \\ a' \\ b' \\ c' \\ e' \end{array} \!\!\begin{array}{c} \begin{array}{ccccc} a' & b' & c' & d' & e' \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

The above shows that

$$A(G) = A(G')$$

Hence $G$ and $G'$ are isomorphic. ■

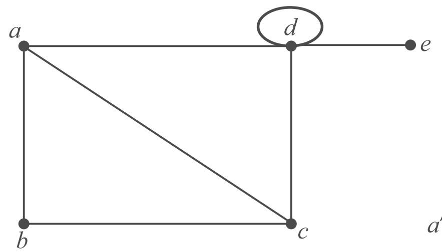
## SUMMARY

1. A **graph** $G$ consists of a nonempty, finite set $V$ of vertices and a set $E$ of edges connecting them: $G = (V, E)$. An edge connecting vertices $u$ and $v$ is denoted by $(u, v)$.
2. A graph with no loops or parallel edge is a **simple graph**.
3. Two vertices $v$ and $w$ in a graph $G$ are **adjacent** if $(v, w)$ is an edge in $G$.
4. The **degree** of a vertex $v$ is the number of edges meeting at $v$.

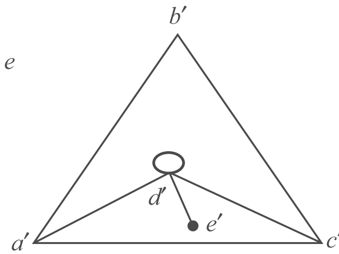
5. The **adjacency matrix** of $G$ is an $n \times n$ matrix $A = (a_{ij})$, where $a_{ij}$ = the number of edges from vertex $v_i$ to vertex $v_{j.}$
6. Let $e$ denote the number of edges of a graph with $n$ vertices $v_1, v_2, ..., v_n$. Then
$$\sum_{i=1}^{n} \deg(v_i) = 2e$$
7. A **Subgraph** of a graph $G = (V, E)$ is a graph $H = (V_1, E_1)$, where $V_1 \subseteq V$ and $E_1 \subseteq E$.
8. A simple graph with $n$ vertices is **complex graph** $K_n$ if every pair of distinct vertices is connected by an edge.
9. Let $G = (V, E)$ be a simple graph such that $V = V_1 \cup V_2$, $V_1 \cup V_2 = \emptyset$, and every edge is incident with a vertex in $V_1$ and $V_2$. $G$ is a **bipartite graph**. If every vertex in $V_1$ is adjacent to every vertex in $V_2$, $G$ is a **complete bipartite graph**. If $|V_1| = m$ and $|V_2| = n$, then $G$ is denoted by $K_{m,n}$.
10. A **weighted graph** is a simple graph in which every edge is assigned a positive number, called the **weight** of the edge
11. An ***r*-regular graph** is a simple graph in which every vertex has the same degree $r$
12. The **complement** $G' = (V, E')$ of a simple graph $G = (V, E)$ contains all vertices in $G$. An edge $(u,v) \in E'$ if and only if $\{u, v\} \notin E$.
13. Two simple graphs, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, are **isomorphic** if a bijection $f : V_1 \rightarrow V_2$, exists such that $\{u, v\} \in E_1$ if and only if $\{f(u), f(v)\} \in E_2$. The function $f$ is an **isomorphism** between $G_1$ and $G_2$. If $G_1$ and $G_2$ are isomorphic, they have exactly the same properties.
14. An **isomorphism invariant** is a property shared by isomorphic graphs.
15. A **path of length** $n$ from a vertex $v_0$ to a vertex $v_n$ is a sequence of vertices $v_i$ and edges $e_i$ of the form $v_0 - e_1 - v_1 \dots e_n - v_n$, where each edge $e_i$ is incident with the vertices $v_i - 1$ and $v_i$ $(1 \le i \le n)$. A **simple** path contains no repeated vertices, except perhaps the endpoints.
16. A path from $v_0$ to $v_n$ is **closed** if $v_0 = v_n$. Otherwise, it is **open.**
17. A **cycle** is a simple closed path.
18. A **circuit** is a simple closed path with no repeated edges.
19. A **connected graph** contains a path between any two distinct vertices.
20. The length of a simple path between any two vertices of a connected graph is at most $n - 1$.
21. If $A$ is the adjacency matrix of a connected graph, the number of paths of length $k$ from vertex $v_i$ to $v_j$ is given by $ij$th entry of $A^k$, where $1 \le k \le n - 1$.
22. A simple path in connected graph is **Hamiltonian** if it contains every vertex.
23. A cycle in a connected graph is **Hamiltonian** if it contains every vertex.
24. A connected graph that contains a Hamiltonian cycle is **Hamiltonian.**

## EXERCISES

1. How many vertices are needed to construct a graph with 6 edges in which each vertex is of degree? **[Ans.** 6**]**
2. Show that the degree of a vertex of a simple graph $G$ on $n$-vertices cannot exceed $n - 1$
3. What is the size of an $r$-regular $(p, q)$ graph? **[Ans.** $q = \dfrac{p \times r}{2}$ **]**
4. Show that the following graphs are isomorphic.




(G)




(G)′

5. Let $Q = (V, E, \phi)$ be the graph where
$V = \{A, B, C, D, E, F, G, H\}$, $E = \{a, b, c, d, e, f, g, h, i, j, k, l\}$
and

$$\phi = \begin{pmatrix} a & b & c & d & e & f & g & h & i & j & k & l \\ A & A & D & E & A & E & B & F & G & C & A & E \\ B & D & E & B & B & G & F & G & C & C & A & G \end{pmatrix}$$

What is the degree sequence of $Q$?

6. Consider the following unlabeled pictorial representation of $Q$


(*a*) Create a pictorial representation of $Q$ by labeling $P'(Q)$ with the edges and vertices of $Q$.

(*b*) A necessary condition that a pictorial representation of a graph $R$ can be created by labeling $P'(Q)$ with the vertices and edges of $R$ is that the degree sequence of $R$ be (0, 2, 2, 3, 4, 4, 4, 5).
True or false? Explain.

(*c*) A sufficient condition that a pictorial representation of a graph $R$ can be created by labeling $P'(Q)$ with the vertices and edges of $R$ is that the degree sequence of $R$ be (0, 2, 2, 3, 4, 4, 5). True or false? Explain.

7. In each of the following problems information about the degree sequence of a graph is given. In each case, decide if a graph satisfying the specified conditions exists on not. Give reason in each case.

(*a*) A graph $Q$ with degree sequence (1, 1, 2, 3, 3, 5)?

(*b*) A graph $Q$ with degree sequence (1, 2, 2, 3, 3, 5), loops and parallel edges allowed?

(*c*) A graph $Q$ with degree sequence (1, 2, 2, 3, 3, 5), no but parallel edges allowed?

(*d*) A graph $Q$ with degree sequence (1, 2, 2, 3, 3, 5), no loops or parallel edges allowed?

(*e*) A simple graph $Q$ with degree sequence (3, 3, 3, 3)?

(*f*) A graph $Q$ with degree sequence (3, 3, 3, 3), no loops or parallel edges allowed?

(*g*) A graph $Q$ with degree sequence (4, 4, 4, 4, 4), no loops or parallel edges allowed?

(*h*) A graph $Q$ with degree sequence (4, 4, 4, 4, 4), no loops or parallel edges allowed?

(*i*) A graph $Q$ with degree sequence (4, 4, 4, 4, 6), no loops or parallel edges allowed?

**8.** Divide the following graphs into isomorphism equivalence classes and justify your answer; *i.e.*, explain why you have the classes that you do. In all cases $V = 4$.

(*a*) $\phi = \begin{pmatrix} a & b & c & d & e & f \\ \{1,2\} & \{1,2\} & \{2,3\} & \{3,4\} & \{1,4\} & \{2,4\} \end{pmatrix}$

(*b*) $\phi = \begin{pmatrix} A & B & C & D & E & F \\ \{1,2\} & \{1,4\} & \{1,4\} & \{1,2\} & \{2,3\} & \{3,4\} \end{pmatrix}$

(*c*) $\phi = \begin{pmatrix} u & v & w & x & y & z \\ \{2,3\} & \{1,3\} & \{3,4\} & \{1,4\} & \{1,2\} & \{1,2\} \end{pmatrix}$

(*d*) $\phi = \begin{pmatrix} P & Q & R & S & T & U \\ \{3,4\} & \{2,4\} & \{1,3\} & \{3,4\} & \{1,2\} & \{1,2\} \end{pmatrix}$

**9.** A graph $G = (V, E)$ is called bipartite if $V$ can be partitioned into two sets $C$ and $S$ such that each edge has one vertex in $C$ and one vertex in $S$. As a specific example, let $C$ be the set of courses at the university and $S$ the set of students. Let $V = C \cup S$ and let $\{s, c\} \in E$ if and only if student s is enrolled in course $c$.

(*a*) Prove that $G = (V, E)$ is a simple graph.

(*b*) Prove that every cycle of $G$ has an even number of edges.

**10.** An *oriented simple graph* is a simple graph which has been converted to a digraph by assigning an orientation to each edge. The orientation of $\{u, v\}$ can be thought of as a mapping of it to either $(u, v)$ or $(v, u)$.

(*a*) Give an example of a simple digraph that has no loops but is not an oriented simple graph.

(*b*) Find the number of oriented simple digraphs.

(*c*) Find the number of them with exactly $q$ edges.

**11.** A binary relation $R$ on $S$ is an order relation if it is reflexive, antisymmetric, and transitive. $R$ is antisymmetric if for $(x, y) \notin R$ with $x \neq y$, $(y, x) \in R$. Given an order relation $R$, the covering relation $H$ of $R$ consists of all $(x, z) \in R$, $x \neq z$, such that there is no $y$, distinct from both $x$ and $z$, such that $(x, y) \in R$ and $(y, z) \in R$. A pictorial representation of the covering relation as a directed graph is called a "Hasse diagram" of $H$.

(*a*) Show that the divides relation on

$S = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$

Is an order relation. By definition, $(x, y)$ is in the divides relation on $S$ is $x$ is a factor of $y$. Thus, (4, 12) is in the divides relation. $x| y$ is the standard notation for $x$ is a factor of $y$.

(*b*) Find and draw a picture of the directed graph of the covering relation of the divides relation.

***Hint:*** You must find all pairs $(x, z) \in S \times S$ such that $x| y$ but there does not exist any 4, $x < y < z$, such that $x| y$ and $y|z$.

## Suggested Readings


1. **N. Cristofides**, *Graph Theory*: *An Algorithmic Approach*, Academic Press, New York, 1975.
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3. **J.A. McHugh**, *Algorithmic Graph Theory*, Prentice-Hall, Englewood Cliffs, NJ, 1990.
4. **A. Ralston**, "Debrujin Sequences—A model Example of the Interaction of Discrete Mathematics and Computer Science," *Mathematics Magazines,* Vol. 55 (May 1982), pp. 131-143.
5. **S.S. Skieno**, *Implementing Discrete Mathematics,* Addison-Wesley, Reading, MA, 1990.
6. **K. Thulasiraman** and **M.N.S. Swamy**, *Graphs*: *Theory and Algorithms,* Wiley, New York, 1992.


❑❑❑

# 2 Trees

**Gustav Robert Kirchoff (1824–1887)**

Gustav Robert Kirchoff (1824-1887) was a German physicist born in Konigsberg, Prussia. His father was a lawyer. He graduated in early age of 18 from the local gymnasium from the university of Konigsberg and received his doctorate five years later. He started teaching in Berlin in 1848 and two years later he joined as a faculty at the university of Breslan where he met well known Chemist Robert Bunsen. In 1854 both moved to Heidelberg where he made the greatest contribution to science (Kirchoff's laws)

In 1875, Kirchoff accepted the chair of theoretical physics at the university of Berlin which he hold until his death.

Kirchoff made significant contributions to every branch of physics and engineering. Some of his results are widely used in data structure and graph theory.

## 2.1 Introduction

The word "tree" suggest branching out from a root and never completing a cycle. As a graph, trees have many applications especially in data storage, searching, and communication. Trees play an important role in a variety of algorithms. We use decision trees to enhance our understanding of recursion. We can point out some of its applications to simple situations and puzzles and games, deferring the applications to more complex scientific problems.

*Kirchhoff* (1824–1887) developed the theory of trees in 1847, in order to solve the system of simultaneous linear equations which give the current in each branch and around each circuit of an electric network. In 1857, *Cayley* discovered the important class of graphs called trees by considering the changes of variables in the differential calculates. Later on, he was engaged in enumerating the isomers of saturated hydro-carbons ($C_nH_{2n+2}$).



S. K. Yadav, *Advanced Graph Theory*,
https://doi.org/10.1007/978-3-031-22562-8_2

An acyclic graph, one not containing any cycle, is called a forest. A connected forest is called a tree. Thus, a forest is a graph whose components are trees. The vertices of degree one in a tree are its leaves. Every non trivial tree has atleast two leaves-take, for example, the ends of a longest path. This little fact often comes in handy, especially in induction proofs about trees: If we remove a leaf from a tree, what remains is still a tree (see Fig. 2.1)

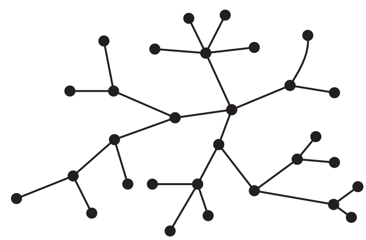


**Fig. 2.1: A Tree**

## 2.2 Definitions of Tree

If $G$ is a connected graph without any cycles then $G$ is called a tree. A tree is also called a free tree. If $|V| = 1$, then $G$ is connected and hence it is a tree.

### *Theorem 2.1*

*Alternative definitions of a tree.*

*If G is a connected graph, the following are equivalent:*

($i$) *G is a tree.*

($ii$) *G has no cycles.*

($iii$) *For every pair of vertices $u \neq v$, there is exactly one path from u to v.*

($iv$) *Removing any edge from G gives a graph which is not connected.*

($v$) *The number vertices of G is one more than the number of edges of G.*

### *Proof:*

We are given that, $G$ is connected, thus by definition of a tree ($i$) and ($ii$) are equivalent we have that two vertices $u \neq v$ are on a cycle of $G$ iff there are atleast two paths from $u$ to $v$ that have no vertices in common except the end points $u$ and $v$. from the above statement ($ii$) $\Rightarrow$ ($iii$)

If $\{u, v\}$ is an edge, it follows from ($iii$) that the edge is only path from $u$ to $v$ and removing it disconnects the graph have ($iii$) $\Rightarrow$ ($iv$).

We have seen that ($i$) and ($ii$) are equivalent, and we also have seen that ($ii$) $\Rightarrow$ ($iii$) $\Rightarrow$ ($iv$) $\Rightarrow$ ($v$),

so ($i$), ($ii$), ($iii$) and ($iv$) are equivalent. All that remains is to include ($v$) in this equivalence class of statements. ∎

## 2.3 Forest

A forest is a graph all of whose connected components are trees. In particular, *a forest with single component is a tree.*

■ **Example 2.1:** *A Relation for Forests*

Suppose a forest has $v$ vertices, $e$ edges and $c$ (connected) components. What values are possible for the triple of numbers $(v, e, c)$? It might seem at first that almost anything is possible, but this is not so. In fact $v - c = e$. Why? Let the forest consist of trees $T_1, ..., T_c$ and let the triples for $T_i$ be $(v_i, e_i, c_i)$. Since a tree is connected, $C_i = 1$. By the theorem, $e_i = v_i - 1$. Since $v = v_1 + ...+ v_c$ and $e = e_1 + ,..., + e_c$ we have

$$e = (v_1 - 1) + (v_2 - 1) + ... + (v_c - 1)$$
$$= (v_1 +...+ v_c) - c = v - c.$$

Suppose a forest has $e = 12$ and $v = 15$. We know immediately that it must be made up of three trees because $c = v - e = 15 - 12$.

Suppose we know that a graph $G = (V, E, \phi)$ has $v = 15$ and $c = 3$, what is the fewest edges it could have? For each component of G, we can remove edges one by one until we cannot remove any more without breaking the component into two components. At this point, we are left with each component a tree. Thus we are left with a forest of $c = 3$ trees that still has $v = 15$ vertices. By our relation $v - c = e$, this forest has 12 edges. Since we may have removed edges from the original graph to get to this forest, the original graph has at least 12 edges.

What is the maximum number of edges that a graph $G = (V, E, \phi)$ with $v = 15$ and $c = 3$ could have? Since we allow multiple edges, a graph could have an arbitrarily large number of edges for a fixed $v$ and $c$. if $e$ is an edge with $\phi(e) = \{u, v\}$, add in as many edges $e_i$ with $\phi(e_i) = \{u, v\}$ as we wish. Hence we will have to insist that $G$ be a simple graph.

What is the maximum number of edges that a simple graph $G$ with $v = 15$ and $c = 3$ could have? We start with a graph where $c$ is not specified. The edges in a simple graph are a subset of $P_2(V)$ and since $P_2(V)$ has $\binom{V}{2}$ elements, a simple graph with $v$ vertices has at most $\binom{V}{2}$ edges.

We return to the case when we know there must be three components in our simple graph. Suppose the number of vertices in the components are $v_1$, $v_2$, and $v_3$. Since there are no edges between components, we can look at each component by itself. Using the result in the previous paragraph for each component, the maximum number of possible edges is $\binom{v_1}{2} + \binom{v_2}{2} + \binom{v_3}{2}$. We don't know $v_1$, $v_2$,

$v_3$. All we know is that they are strictly positive integers that sum to $v$. It turns out that the maximum occurs when one of $v_i$ is as large as possible and the others equal 1, but the proof is beyond this course. Thus the answer is $\binom{v-2}{2}$, which in our case is $\binom{13}{2} = 78$. In general, if there were $c$ components, $c - 1$ components would have one vertex each and the remaining component would have $v - (c - 1) = v + 1 - c$ vertices. Hence there can be no more than $\binom{v+1-c}{2}$ edges.

The above results can conclude that:

(*i*) There is no graph $G = (V, E, \phi)$ with $v - c > e$

(*ii*) If $v - c = e$, the graph is a forest of c trees and any such forest will do as an example.

(*iii*) If $v - c < e$, there are many examples, none of which are forests.

(*iv*) If $v - c < e$ and we have a simple graph, then we must have $e \leq \binom{v+1-c}{2}$. ■

## 2.4 Rooted Graph

A pair $(G, v)$, consisting of a graph $G = (V, E, \phi)$ and a specified vertex $v$, is called a rooted graph with root $v$.

A rooted tree $T$ with the vertex set $V$ is the tree that can be defined recursively as:

(*i*) $T$ has a specially designated vertex $v_1 \in V$, called the root of $T$. The subgraph of $T_1$ consisting of the vertices $V - \{v\}$ is partitionade into subgraphs.

(*ii*) $T_1, T_2, \ldots T_r$ each of which is itself a rooted tree. Each one of these $r$-rooted tree is called a subtree of $v_1$ (as shown in fig 2.2)

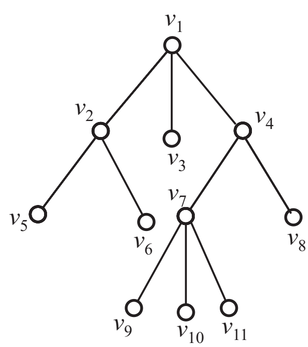



**Fig. 2.2: A Rooted Tree**

***Note:*** *A tree $T^*$ of a spanning tree $T$ in a connected graph $G$ is the spanning subgraph of $G$ containing exactly those edges of $G$ which are not in $T$. The edges of $G$ which are not in $T^*$ are called its twigs.*

## 2.5 Parent, Child, Sibling and Leaf

Let $(T, r)$ be a rooted tree. If $w$ is any vertex other than $r$, let $r = v_0, v_1, v_2, \ldots, v_k, v_{k+1} = w$, be the list of vertices on the unique path from $r$ to $w$. We can say $v_k$ is parent of $w$ and $w$ a child of $v_k$ parents and children are also called father and sons. Vertices with the same parent are siblings. A vertex with no children is called a leaf. All the other vertices are internal vertices of the tree.

## 2.6 Rooted Plane Tree

Let $(T, r)$ be a rooted tree. For each vertex, order the children of the vertex. The result is a rooted plane tree, which is abbreviated as *RP*-tree. *RP*-trees are also called ordered trees. An *RP*-tree is also called, a decision tree, and, when there is no chance of misunderstanding, simply a tree.

■ **Example 2.2***:* *A Rooted Plane Tree*

Fig. 2.3 is a picture of a rooted plane tree $T = (V, E, \phi)$. In this situation $V$ = 11 and $E = \{a, b, c, d, e, f, g, h, i, j\}$. There are no parallel edges or loops. The root is $r = 1$. For each vertex, there is a unique path from the root to that vertex. Since $\phi$ is an injection, once $\phi$ has defined that unique path can be specified by the vertex sequence alone. The path from the root is to 6 is (1, 3, 6). The path from the root to 9 is (1, 3, 6, 9) (In computer science we refer to the path from the root to vertex $v$ as the "stack" if $v$.

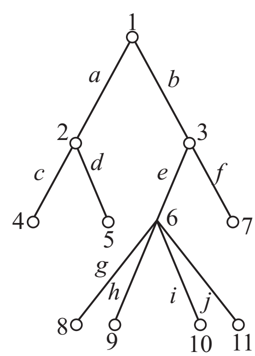



**Fig. 2.3: RP – Tree**

The vertex 6 is the parent of the vertex 9. The vertices 8, 9, 10 and 11 are the children of 6 and, they are siblings of each other. The leaves of the tree are 4, 5, 7, 8, 9, 10, 11. All other vertices (including root) are the internal vertices of the tree.

We must remember that, an *RP*-tree is a tree with added properties. So, we must satisfy, $T$ has no cycle. There is a unique path between any two vertices. Removing any edge gives a graph which is not connected. (e.g. removing $j$ disconnects $T$ into a tree with 10 vertices and a tree with 1 vertices; removing

e disconnects $T$ into a tree with 6 vertices and one with 5 vertices). At last, the number of edges (10) is one less than the number of vertices. ■

■ **Example 2.3:** *Traversing a Rooted Plane Tree*

As in case of decision trees, we can define the notion of depth first traversals of a rooted plane tree. We can imagine going around the *RP*-tree (Fig. 2.3) following arrows. We can start at the root, 1, go down edge a to vertex 2 etc. We find the sequence of vertices as encountered in this process: 1, 2, 4, 2, 5, 2, 1, 3, 6, 8, 6, 9, 6, 10, 6, 11, 6, 3, 7, 3, 1. This sequence of vertices is known as the depth first vertex sequence, DFV($T$), of the rooted plane Tree $T$. The number of times each vertex appears in $DFV(T)$ is one plus the number of children of that vertex. For edges, the corresponding sequence is $a$, $c$, $c$, $d$, $d$, $a$, $b$, $e$, $g$, $g$, $h$, $h$, $i$, $i$, $j$, $j$, $e$, $f$, $f$, $b$. This sequence is the depth first edge sequence, $DFE(T)$, of the tree. Every edge appears exactly twice in $DFE(T)$. If the vertices of the rooted plane tree are read left to right, top to bottom, we can obtain the sequence 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11. This is called the breadth first vertex sequence, $BFV(T)$. Similarly, the breadth first edge sequence, $BFE(T)$ is $a$, $b$, $c$, $d$, $e$, $f$, $g$, $h$, $i$, $j$. (As shown in Fig 2.4).

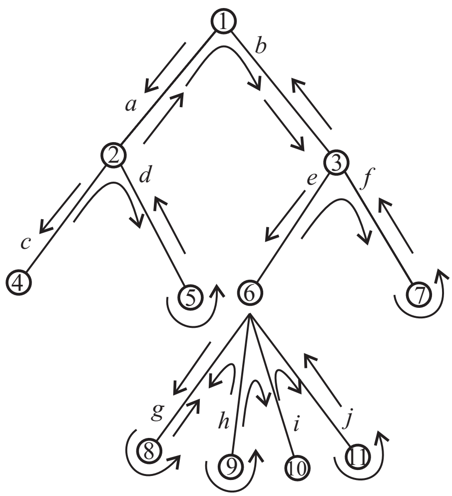



**Fig. 2.4: TRP – Tree**

The sequence $BFV(T)$ and $BFE(T)$ are linear ordering of the vertices and edges of the *RP*-tree $T$ (each vertex or edge appears exactly once in the sequence). We also associate linear ordering with $DFV(T)$ which is called the sequence of vertices of $T$.

$PREV(T)$, and the post order sequence of vertices of $T$, $POSV(T)$.

Here $PREV(T)$ = 1, 2, 4, 5, 3, 6, 8, 9, 10, 11, 7 is the sequence of first occurrances of the vertices of $T$ in $DFV(T)$.

$POSV(T) = 4, 5, 2, 8, 9, 10, 11, 6, 7, 3, 1$ is the sequence of last occurrences of the vertices of $T$ in $DFV(T)$.

We can notice that the order in which the leaves of $T$ appear, 4, 5, 8, 9, 10, 11, is the same in both $PREV(T)$, and $POSV(T)$. ■

### *Theorem 2.2*

*A (p, q) graph is a tree iff it is acyclic and $p = q + 1$ or $q = p - 1$.*

### *Proof:*

If $G$ is a tree, then it is acyclic. From definition, to verify the equation $p = q + 1$.

We can employ inductions on $p$.

for $p = 1$, the result is so trivial.

We can assume, then that the equality $p = q + 1$ holds for all $(p, q)$ trees with $p \geq 1$ vertices

Let $G_1$ be a tree with $p + 1$ vertices

Let $v$ be an end vertex of $G_1$.

The graph $G_2 = G_1 - v$ is a tree of order $p$, and so

$$p = |E(G_2)| + 1$$

Since $G_1$ has one more vertex and one more edge than that of $G_2$.

$$|V(G_1)| = p + 1 = |E(G_2)| + 1 + 1 = |E(G_1)| + 1.$$

$\therefore$ $|V(G_1)| = |E(G_1)| + 1$. (As shown in Fig. 2.5)

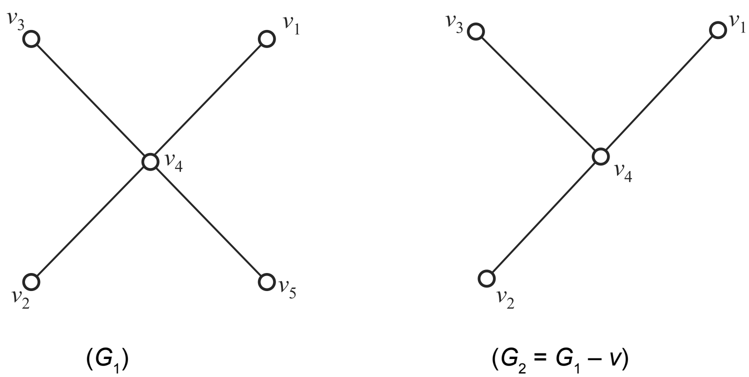



**Fig. 2.5**

Converse: Let $G$ be an acyclic $(p, q)$ graph with $p = q + 1$

We can show $G$ is a tree. To do so, we need only to verify that $G$ is connected. We can denote by $G_1, G_2, \ldots, G_k$ the components of $G$, where $k \geq 1$.

Further, Let $G_i$ be a $(p_i, q_i)$ graph

Since each $G_i$ is a tree, $p_i = q_i + 1$.

Hence $$p - 1 = q = \sum_{i=1}^{k} q_i$$
$$= \sum_{i=1}^{k} q_i\,(p_i - 1) = p - k$$

$\Rightarrow$ $p - 1 = p - k$

$\Rightarrow$ $k = 1$. and $G$ is connected.

Hence, $(p, q)$ graph is tree. ■

### *Theorem 2.3*

*A $(p, q)$ graph G is a tree iff G is connected and $p = q + 1$.*

### *Proof:*

Let $G$ be a $(p, q)$ tree. From definition of $G$, it is connected and $A(p, q)$ graph is a tree iff it is a acyclic and $p = q + 1$, or $q = p - 1$ (Theorem 2.2).

Conversely, we assume $G$ is connected $(p, q)$ graphs with $p = q + 1$

It is sufficient to show that $G$ is acyclic.

If $G$ contains a cycle $C$ and $e$ is an edge of $C$, then $G - e$ is a connected graphs with $p$ vertices having $p - 2$ edges.

It is impossible by definition and contridicts our assumption.

Hence $G$ is connected. ■

■ **Example 2.4:** *A tree has five vertices of degree 2, three vertices of degree 3 and four vertices of degree 4. How many vertices of degree 1 does it have?*

### *Solution:*

Let $x$ be the number of nodes of degree 1.

∴ Total number of vertices

$$= 5 + 3 + 4 + x$$
$$= 12 + x.$$

The total degree of tree $= 5 \times 2 + 3 \times 3 + 4 \times 4 + x$

$$= 35 + x$$

The number of edge in the tree is half of the total degree of it.

If $G = (V, E)$ be the tree, then, we have

$$|V| = 12 + x \text{ and } |E| = \frac{35 + x}{2}$$

If any tree $|E| = |V| - 1.$

$$\Rightarrow \qquad \frac{35+x}{2} = 12 + x - 1$$

$$\Rightarrow \qquad \frac{35+x}{2} = 24 + 2x - 2$$

$$\Rightarrow \qquad x = 13.$$

∴ There are 13 nodes of degree one in the tree. ■

■ **Example 2.5:** *The Number of Labeled Trees*

*How many n-vertex labeled trees are there? Or, count the number of trees with vertex set V =* $\underline{n}$*. The answer has been obtained in a variety of ways.*

### *Solution:*

Suppose $f$ is a function from $V$ to $V$. We can represent this as a simple digraph $(V, E)$ where the edges are $\{(v, f(v)) \mid v \in V\}$. The function

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 1 & 10 & 9 & 2 & 8 & 2 & 2 & 5 & 1 & 6 & 11 \end{pmatrix}$$

corresponds to the directed graph

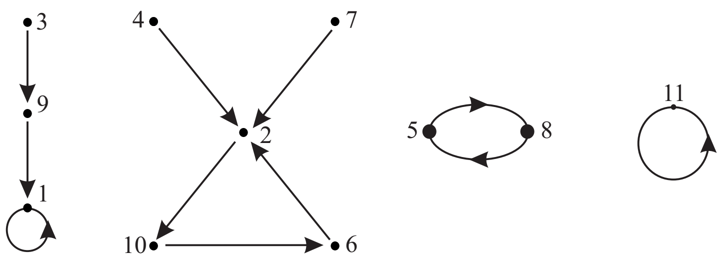



**Fig. 2.6: Functional Digraph**

Such graphs are called *functional digraphs.* We should be able to convince ourself that a functional digraph consists of cycles (including loops) with each vertex on a cycle being the root of a tree of noncyclic edges. The edges of the trees are directed toward the roots. In the Fig. 2.6.

- 1 is the root of the tree with vertex set {1, 3, 9},
- 2 is the root of the tree with vertex set {2, 4, 7}
- 5 is the root of the tree with vertex set {5},
- 6 is the root of the tree with vertex set {6},
- 8 is the root of the tree with vertex set {8}
- 10 is the root of the tree with vertex set {10} and
- 11 is the root of the tree with vertex set {11}.

In a tree, there is a unique path from the vertex 1 to the vertex $n$. Remove all the edges on the path and list the vertices on the path, excluding 1 and $n$, in the order they are encountered. Interpret this list as a permutation in 1 line form. Draw the functional digraph for the cycle form, adding the cycles form, adding the cycles (1) and ($n$). Add the trees that are attached to each of the cycle vertices, directing their edges toward the cycle vertices. Consider the following Fig. 2.7.

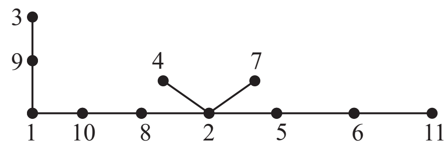



**Fig. 2.7: Tree with Cycle Certices**

The one line form is 10, 8, 2, 5, 6. In two line form it is. Thus the cycle form is (2, 10, 6) (5, 8). When we add the two cycles (1) and (11) to this, draw the directed graph, and attach the directed trees, we obtain the functional digraph pictured earlier.

This gives us a one-to-one correspondence between trees with $V = \underline{n}$ and functions $f : \underline{n} \to \underline{n}$ with $f(1) = 1$ and $f(n) = n$. In creating such a function, there are n choices for each of $f(2), ..., f(n - 1)$. Thus there are $n^{n-2}$ such functions and hence $n^{n-2}$ trees. ■

■ **Example 2.6:** *A tree has 2n vertices of degree 1, 3n vertices of degree 2 and n vertices of degree 3. Determine the number of vertices and edges in the tree.*

***Solution:***

We are given that total number of vertices in the tree is $2n + 3n + n = 6n$

The total degree of the tree is $2n \times 1 + 3n \times 2 + n \times 3$

$$= 11n.$$

The number of edges in the tree will be half of $11n$.

If $G = (V, E)$ be a tree then, we have

$$|V| = 6n \text{ and } |E| = \frac{11n}{2}$$

for a tree $\quad |E| = |V| - 1$

We have, $\quad \dfrac{11n}{2} = 6n - 1.$

$\Rightarrow \quad 11n = 12n - 2$

$\Rightarrow \quad n = 2$

$\Rightarrow \quad$ nodes = 12, edges = 11 ■

## 2.7 Binary Trees

A binary tree is a rooted tree where each vertex $v$ has atmost two subtrees; if both subtrees are present, one is called a left subtree of $v$ and the other is called right subtree of $u$. Iff only one subtree is present, it can be designated either as the left subtree or right subtree of $v$.

OR

A binary tree is a 2-ary tree in which each child is designated as a left child or right child.

In a binary tree every vertex has two children on no children.

A binary tree has the following properties:

(*i*) The number of vertices $n$ in a complete binary tree is always odd. This is because there is exactly one vertex of even degree, and remaining $(n - 1)$ vertices are of odd degree. Since from the theorem (number of vertices of odd degree is even) $n - 1$ is even. Hence n is odd.

(*ii*) Let $p$ be the number of end vertices in a binary tree $T$. Then $n - p - 1$ is the number of vertices of degree 3. The number of edges in $T$ is

$$\frac{1}{2}[p + 3(n - p - 1) + 2] = n - 1.$$

on $$p = \frac{n+1}{2}$$

(*iii*) A non-end vertex in a binary tree is called an internal vertex. The number of internal vertices in a binary tree is one less than the number of end vertices.

(*iv*) In a binary tree, a vertex $v_i$ is said to be at level $l_i$ and $v_i$ is at a distance $l_i$ from the root. Thus the root is at level $O$.

(*v*) The maximum number of vertices possible in a $k$ = level binary tree is $2^0 + 2^1 + 2^2 + 2^3 + ,\ldots, + 2^k \geq n$.

(*vi*) The maximum level $I_{max}$ of any vertex in a binary tree is called the *height* of the tree.

(*vii*) To construct a binary tree for a given n s.t. the farthest vertex is as far as possible from the root, we must have exactly two vertices at each level, except at the O level.

Hence max. $I_{max}. = \dfrac{n-1}{2}$.

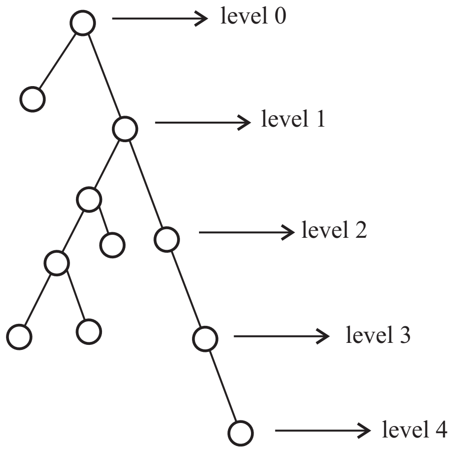



**Fig. 2.8: 13-vertices, 4-level Binary Tree**

(*viii*) The minimum possible height of $n$-vertex binary tree is min $I_{max}$ = [$log_2$ $(n + 1) - 1$]

(*ix*) The sum of levels of all end vertices is called *path length* of a tree.

### *Theorem 2.4*

*There are at the most $n^h$ leaves in an n-ary tree of height h.*

### *Proof:*

The theorem can be proved by the principle of mathematical induction on the height of the tree.

As basic step to be $h = 0$ *i.e.* tree consists of root node only.

Since $n^0 = 1$, the basis step is tree.

We assume that the above statement is tree for $h = k$.

*i.e.* an $n$-ary tree of height $k$ has at the most $n^k$ leaves.

If we add $n$ nodes to each of the leaf node of $n$-ary tree of height $k$, the total number of leaf nodes will be the most

$$n^h \times n = n^{h+1}$$

Hence inductive step is also true

This proves the statement for all $h \geq 0$. ■

### *Theorem 2.5*

*In a complete n-ary tree with m internal nodes, the number of leaf node l is given by the formula* $l = \dfrac{(n-1)(x-1)}{n}$

*Where x is the total number of nodes in the tree.*

***Proof:***

Given that, the tree has $m$ internal nodes and it is complete $n$-ary, so total number of nodes.

$$x = n \times m + 1.$$

Then, we have, $$m = \frac{x-1}{n}$$

It is also given that $l$ is the number of leaf nodes in the tree. We have, $x = m + l + 1$.

Substituting the value of $m$ in this equation, we get

$$x = \left(\frac{x-1}{n}\right) + (l+1)$$

$$\Rightarrow \qquad l = \frac{(n-1)(x-1)}{n} \qquad \blacksquare$$

## 2.8 Spanning Trees

A spanning tree of a simple graph $G = (V, E)$ is a subgraph $T = (V, E')$ which is a tree and has the same set of vertices as $G$.

### *Connected graphs and Spanning Trees*

Since a tree is connected, a graph with a spanning tree must be connected. On the other hand, it is not hard to see that every connected graph has a spanning tree. Any simple graph $G = (V, E)$ has a subgraph that is a tree, $T' = (V', E')$. Take $V' = \{v\}$ to be one vertex and $E'$ empty. Suppose that $T' = (V', E')$ is the largest such "subtree". If $T'$ is not a spanning tree then there is a vertex $w$ of $G$ that is not a vertex of $T'$. If $G$ is connected. We choose a vertex $u$ in $T'$ and a path $w = x_1, x_2,\ldots, e_k = u$ from $w$ to $u$. Let $j$, $1 < j \leq k$, be the first integer such that $x_j$ is a vertex of $T'$. Then adding the edge $\{x_j - 1, x_j\}$ and the vertex $x_{j-1}$ to $T'$ creates a subtree $T$ of $G$ that is larger than $T'$, a contradiction of the maximality of $T'$. We have, in fact, shown that a graph is connected if and only if every maximal subtree is a spanning tree. Thus we have: *A graph is connected if and only if it has a spanning tree.* It follows that, if we had an algorithm that was guaranteed to find a spanning tree whenever such a tree exists, then this algorithm could be used to decide if a graph is connected.

### *Weight in a Graph*

Let $G = (V, E)$ be a simple graph and let $\lambda(e)$ a function from $E$ to the positive real numbers. We call $\lambda(e)$ the weight of the edge. If $H = (V', E')$ is a subgraph of $G$, then $\lambda(H)$, the weight of $H$, is the sum of $\lambda(e')$ over all $e' \in E'$.

A minimum weight spanning tree for a connected graph $G$ is a spanning tree such that $\lambda(T) \leq \lambda(T')$ whenever $T'$ is another spanning tree.

### *Theorem 2.6*

*Each connected graph has a spanning tree, i.e. a spanning graph is a tree.*

### *Proof:*

Let $H \subseteq G$ be a minimal connected spanning subgraph, *i.e.* a connected spanning subgraph of $G$ s.t. $H - e$ is disconnected for all $e \in E_H$. Such a subgraph is obtained from $G$ by removing nonbridges.

- To start with, let $H_0 = G$.
- For $i \geq$ o, let $H_{i+1} = H_i - e_i$ where $e_i$ is not a bridge of $H_i$. Since $e_i$ is not a bridge $H_{i+1}$ is a connected spanning subgraph of $H_i$ and thus of $G$.
- $H = H_k$ when only bridges are left.

Hence $H$ is a tree. ■

### *Theorem 2.7*

*A nondirected graphs G is connected iff G containing a spanning tree. Indeed, if we successive –ly delete edges of cycles until no further cycles remains, then the result is a spanning tree of G.*

### *Proof:*

If $G$ has a spanning tree $T$, there is a path between any pair of vertices in $G$ along the tree $T$.

Thus $G$ is connected.

Conversely, We can prove that a connected graph $G$ has a spanning tree by *PMI* on the number $k$ of cycles in $G$. If $k = 0$, then $G$ is connected with no cycles and hence $G$ is already a tree. Suppose that all connected graphs with fewer than $k$ cycles have a spanning tree. Then suppose that $G$ is a connected graph with $k$ cycles. Remove an edge $e$ from one of the cycles. Then $G - e$ is still connected and has a spanning tree by the inductive hypothesis because $G - e$ has fewer cycles then $G$. But since $G - e$ has all the vertices of $G$, the spanning tree for $G - e$ is also one for $G$. ■

## 2.9 Breadth – First Search and Depth – First Search (BFS and DFS)

An algorithm based on the proof of theorem 7 could be designed to produce a spanning tree for a connected graph. If we recall from the proof that all that one need is destroy cycles in the graph by removing an edge from a cycle until

no cycles remain. Unfortunately, such an algorithm not very efficient because it consumes a long time to find cycles. On the other hand, we can define other rather efficient algorithms for finding a spanning tree of a connected graphs. There algorithms are called breadth – first search (BFS) and depth – first search (DFS).

BSF Algorithm for a spanning tree

*Input:* A connected graph $G$ with Vertices $v_1, v_2, \ldots, v_n$.

*Output:* A spanning tree $T$ for $G$.

### *Method*

(*i*) (Start.) Let $v_1$ be the root of $T$. From the set $V = \{v_1\}$.

(*ii*) (Add new edges.) Consider the vertices of $V$ in order consistent with the original labeling. Then for each vertex $x \in V$, add the edge $\{x, v_k\}$. to $T$ where $k$ is the minimum index such that adding the edge $\{x, v_k)$. to $T$ does not produce a cycle. If no edge can be added, then stop; $T$ is a spanning tree for $G$. After all the vertices of $V$ have been considered in order, go to step (*iii*)

(*iii*) (Update $V$.) Replace $V$ by all the children $v$ in $T$ of the vertices $x$ of $V$ where the edges $\{x, v\}$ were added in step (*ii*). Go back and repeat step (*ii*) for the new set $V$.

### *General Procedure*

(*i*) Arbitrarily choose a vertex and designate it as the root. Then add all edges incident to this vertex, such that the addition of edges does not produce any cycle.

(*ii*) The new vertices added at this stage become the vertices at level 1 in a spanning tree, arbitrarily order them.

(*iii*) Next, for each vertex at level 1, visited in order, add each edge incident to this vertex to the tree as long as it does not produce any cycle.

(*iv*) Arbitrarily order the children of each vertex at level 1. This produces the vertices at level 2 in the tree.

(*v*) Continue the same procedure until all the vertices in the tree have been added.

(*vi*) The procedure ends, since there are only a finite number of edges in the graph.

(*vii*) A spanning tree is produced since we have already produced a tree without cycle containing every vertex of the graph.

■ **Example 2.7:** *Use BFS algorithm to find a spanning tree of graph G of Fig. 2.9.*

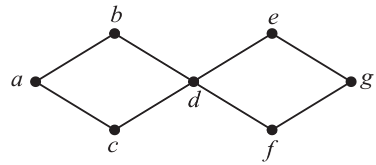


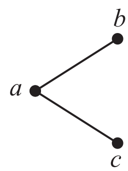

**Fig. 2.9**

***Solution:***

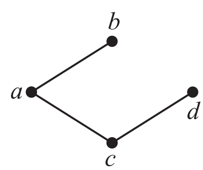
*Step 1:* Choose the vertex $a$ as the root

$$\bullet\ a$$

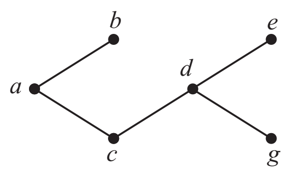
*Step 2:* Add edges incident with all vertices adjacent to $a$, so that edges $\{a, b\}$, $\{a, c\}$ are added. The two vertices $b$ and $c$ are in level 1 in the tree.



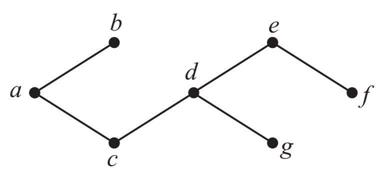

*Step 3:* Add edges from these vertices at level 1 to adjacent vertices not already in the tree. Hence the edge $\{c, d\}$ is to be added. The vertex $d$ is in level 2.




*Step 4:* Add edge from $d$ in level 2 to adjacent vertices not already in the tree. The edge $\{d, e\}$ and $\{d, g\}$ are added. $e$ and $g$ are is level 3.




Step 5: Add edge from $e$ at level 3 to adjacent vertices not already in the tree and hence $\{e, f\}$ is added. The steps of Breath first procedure are obtained.



■ **Example 2.8:** *Illustrate BFS on the graph given in Fig. 2.10.*

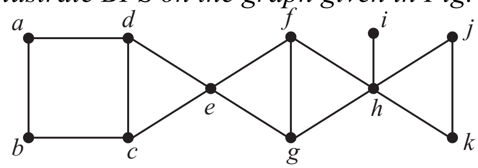



**Fig. 2.10**

### *Solution:*

We firstly select the ordering of the vertices $a, b, c, d, e, f, g, h, i, j, k$.

($i$) Select a as the first vertex in the spanning tree $T$ and designate it as the root of $T$. At this stage, $T$ consists of the single vertex $a$. Add to $T$ all edges $\{a, x\}$ as $x$ sums in order from $b$ to $k$, that do not produce a cycle in $T$.

($ii$) Add $\{a, b\}$ and $\{a, d\}$. These edges are called tree edges for the BFS.

($iii$) Repeat the process for all vertices on level one from the root by examining each vertex in the designated order. Since $b$ and $d$ are at level 1, we first examine $b$.

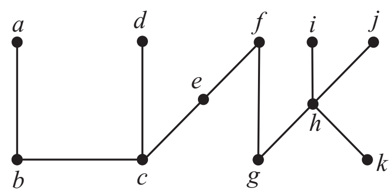
($iv$) For bc, we include the edge $\{b, c\}$ as a tree edge. For $d$, we reject the edge $\{d, c\}$ since inclusion would produce a cycle in $T$. But we include $\{d, e\}$.

($v$) Consider the vertices at level two. Reject the edge $\{c, e\}$; include $\{e, f\}$ and $\{e, g\}$.

($vi$) Repeat the procedure again for vertices on level three. Reject the $\{f, g\}$, but include $\{f, h\}$. At $g$, reject $\{f, g\}$ and $\{g, h\}$.

($vii$) On level four, include $\{h, i\}$, $\{h, j\}$, $\{h, k\}$.

($viii$) We attempt to apply the procedure on level five at $i, j$ and $k$, but no edge can be added at these vertices so the procedure ends.




### *DFS Algorithm for Spanning Tree*

*Input:* A connected graph $G$ with vertices $v_1, v_2, ..., v_n$.

*Output:* A spanning tree $T$ for $G$.

### *Method*

($i$) (Visit a vertex). Let $v_1$ be the root of $T$, and set $L - v_1$ ($L$ is the vertex last visited)

(*ii*) (Find an unexamined edge and an unvisited vertex adjacent to $L$) For all vertices adjacent to $L$, choose the edge $\{L, V_k\}$, where $k$ is the minimum index such that adding $\{L, v_k\}$ to $T$ does not create a cycle. If no such edge exists, go to step (*iii*) otherwise, add edge $\{L, v_k\}$ to $T$ and set $L - v_k$ *i* repeat step (*ii*) at the new value for $L$.

(*iii*) (Back track or terminate). If $x$ is the parent of $L$ in $T$, set $(L - x)$ and apply step (*ii*) at the new value of $L$. If, $L$ has no parent in $T$ (so that $L - v_1$) then the DFS terminates and $T$ is a spanning tree for $G$.

## General Procedure

(*i*) Arbitrarily choose a vertex from the vertices of the graph and designate it as the root.

(*ii*) From a path starting at this vertex by successively adding edges as long as possible where each new edge is incident with the least vertex in the path without producing any cycle.

(*iii*) If the path goes through all vertices of the graph, the tree consisting of this path is a spanning tree. Otherwise, move back to the next to last vertex in the path, and, if possible, form a new path starting at this vertex passing through vertices that were not previously visited.

(*iv*) If this cannot be done, move back another vertex in the path, that is two vertices back in the path and repeat.

(*v*) Repeat this procedure, beginning at the last vertex visited, moving back up and path one vertex at a time, forming new paths that are as long as possible until no more edges can be added.

(*vi*) This process ends since the graph has a finite number of edges and is connected. A spanning tree is produced.

■ **Example 2.9:** *Use DFS to construct a spanning tree of the graph given in Fig. 2.11.*

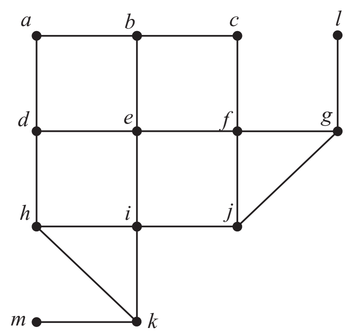
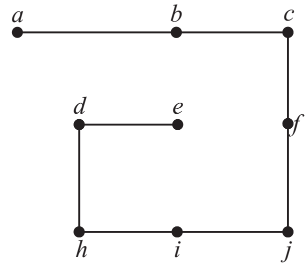



**Fig. 2.11**

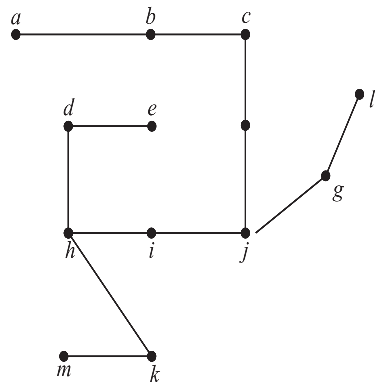
### *Solution:*

(*i*) Start with vertex *a*, build a path by successively adding edges incident with vertices not already in the path, as long as this is possible.

This produces a path $a - b - e - f - g - i - h - d - e$.




(*ii*) Now back track to *d*. There is no path at *d* containing vertices no already visited. So move back track to *h* and form the path $h - k - m$. Now back track to *k*, and *h*, and *i* and *j* then form the path $j - g - l$. This produces the spanning tree.




■

■ **Example 2.10:** *Find a spanning tree of the graph of Fig. 2.12 using DFS algorithm.*

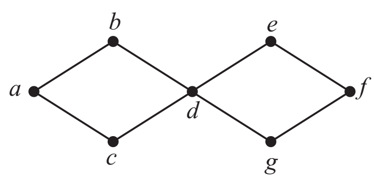



**Fig. 2.12**

### *Solution:*

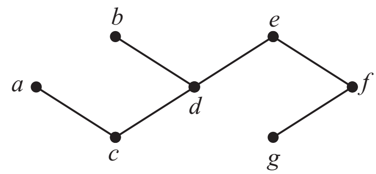
(*i*) Choose the vertex *a*.

(*ii*) Form a path by successively adding edges incident with vertices not already in the path as long as possible.

(*iii*) Now back track of $F$. There is no path beginning at $F$ containing vertices not already visited.

(*iv*) Similarly, after backtrack at $e$, there is no path. So, move back track at $d$ and form the path $d - b$.

This produces the required spanning tree which is shown in Fig. below




■

### *Theorem 2.8*

*For every tree $T = (V, E)$, if $|V| \geq 2$, then $T$ has at least two pendent vertices [Delhi 2006]*

### *Proof:*

Let $|V| = n \geq 2$

Same $|E| = n - 1$, so, if $G = (V, E)$ is an undirected graph or multigraph then $\sum_{v \in V} \deg(v) = 2|E|$

It follows that $2\,(n - 1) = 2|E| = \sum_{v \in V} \deg(v)$

Since $T$ is connected, we have $\deg(v) \geq 1 \;\; \forall \;\; v \in V$.

If $T$ has fewer than two pendent vertices, then either

$$\deg(v) \geq v \in V$$

$$\deg(v^*) = 1 \text{ for only once vertex } v^* \text{ in } V.$$

In the first case we arrive at the contradictions.

$$2(n - 1) = \sum_{v \in V} \deg(v) \geq 2\,|V| = 2n$$

for second case we find that

$$2\,(n - 1) = \sum_{v \in V} \deg(v) \geq 1 + 2(n - 1).$$

Which is another contradiction ■

■ **Example 2.11:** *Let $T = (V, E)$ be a tree with $V = \{v_1, v_2, \ldots, v_n)$ for $n \geq 2$. Prove that the number of pendent vertices in $T$ is equal to* $2 + \sum_{deg(v_i) \geq 3} \left[deg(v_1) - 2\right]$

***Solution:***

Let $1 \leq i\ (< n)$. Let $x_i$ = no. of vertices $v_1$ where $\deg(v) = i$

Then $x_1 + x_2 +,...,+ x_{n-1} = |V| = |E| + 1$.

So, $2|E| = 2(-1 + x_1 + x_2 +,\ldots,+ x_{n-1})$

But $$2|E| = \sum_{v \in V} \deg(v)$$

$$= \{(x_1 + 2x_2 + 3x_3 +,\ldots,+ (n-1)\, x_{n-1}\}$$

***Solving, we get***

$$2(-1 + x_1 + x_2 +,\ldots,+ x_{n-1}) = x_1 + 2x_2 + 3x_3 +,\ldots,+ (n-1)\, x_{n-1}$$

for $x_1$, we find that

$$x_1 = 2 + x_3 + 2x_4 + 3x_5 +,\ldots,+ (n-3)\, x_{n-1}$$

$$= 2 + \sum_{\deg(v_i) \geq 3} \left[\deg(v_i) - 2\right]$$ ■

## 2.10 Minimal Spanning Trees

The application of spanning trees are many and varied, and in order to gain some appreciation for this fact, we would describe what is sometimes called the connector problem. Let us consider that we have a collection of $n$ cities, and that we wish to construct a utility, communication, or transportation network connecting all of the cities. Assuming that we know the cost of building the links between each pair of cities and that, in addition, we wish to construct the network as cheaply as possible. The desired network can be represented by a graph by regarding each city as a vertex and by placing an edge between vertices if a link runs between the two corresponding cities.

Let $G$ be the graph of all possible links between the cities with the non-negative cost of construction $c(e)$ assigned to each edge $e$ in $G$. Then if $H$ is any subgraph of $G$ with edges $e_1, e_2,\ldots, e_m$ the total cost of constructing the network $H$ is

$$C(H) = \sum_{i-1}^{m} C(e_i)$$

A spanning tree $T$ where $c(T)$ is minimal is called a minimal spanning tree of $G$.

### 2.10.1 Kruskal's Algorithm (for Finding a Minimal Spanning Tree)

*Input:* A connected graph $G$ with non negative values assigned to each edge.

*Output:* A minimal spanning tree for $G$.

*Step 1:* Select any edge of minimal value that is not a loop. This is the first edge of $T$.

(If there is more than one edge of minimal value, arbitrarily choose one of these edges).

*Step 2:* Select any remaining edge of $G$ having minimal value that does not for a circuit with the edges already included in $T$.

*Step 3:* Continue step 2 until $T$ contains $n - 1$ edges, where n is the number of vertices of $G$.

### *Theorem 2.9*

Let $G$ be a connected graph where the edges of $G$ are labelled by nonnegative numbers. Let $T$ be an economy tree of $G$ obtained from Kruskal's Algorithm. Then $T$ is a minimal spanning Tree.

### *Proof:*

For each edge e of $G$, let $C(e)$ be the value assigned to the edges by the labelling.

If $G$ has $n$ vertices, an economy tree $T$ must have $(n - 1)$ edges.

Let the edges $e_1, e_2, e_3, \ldots, e_{n-1}$ be chosen in the Krushal's Algorithm.

Then $$C(T) = \sum_{i-1}^{n-1} C(e_i) \qquad \ldots(i)$$

Let $T_O$ be a minimal spanning tree of $G$. We can show that $C(T_O) = C(T)$, and conclude that $T$ is also minimal spanning tree.

If $T$ and $T_0$ are not the same, let $e_i$ be the first edge of $T$ not in $T_0$.

Adding the edge $e_i$ to $T_0$ we obtain the graph $G_0$.

Suppose $e_i = \{a, b\}$. Then a path $P$ from $a$ to $b$ exists in $T_0$ and so $P$ together with $e_i$ produces a circuit $C$ in $G_0$.

Since $T$ contains no circuits, there must be an edge $e_0$ in $C$ that is not in $T$.

The graph $T_1 = G_0 - e_0$ is also a spanning tree of $G$ since $T_1$ has $(n - 1)$ edges.

Moreover,

$$C(T_1) = C(T_0) + C(e_i) - C(e_0)$$

We know that, $C(T_0) \leq C(T_1)$ since $T_0$ was a minimal spanning tree of $G$.

Thus

$$C(T_1) - C(T_0) = C(e_i) - C(e_0) \geq 0.$$

It implies that, $\quad C(e_i) \geq C(e_0)$.

Since $T$ was constructed by Kruskal's Algorithm, $e_i$ is an edge of smallest value that can be added to the edges $e_1, e_2, e_3, \ldots, e_{i-1}$ without producing a circuit.

Also, if $e_0$ is added to the edges $e_1$, $e_2$, $e_3$ ,…, $e_{i-1}$, no circuit is produced because the graph formed is a subgraph of the tree $T_0$.

Therefore, $C(e_i) = C(e_0)$,

so that $C(T_1) = C(T_0)$.

We have constructed from $T_0$ a new minimal spanning tree $T_1$ such that the number of edges common to $T_1$ and $T$ exceeds the number of edges common to $T_0$ and $T$ by one edge, namely $e_i$.

Repeating this procedure, to construct minimal spanning tree $T_2$ with one more edge in common with $T$ than was in common between $T_1$ and $T$.

By containing this process, we finally reach at a minimal spanning tree with all edges in common with $T$, and we conclude that $T$ is a minimal spanning tree. ■

■ **Example 2.12:** *Use Kruskal's Algorithm to find minimal spanning tree for Fig. 2.13. Also find the cost of it.*

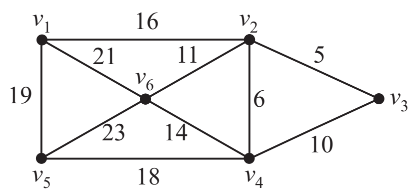


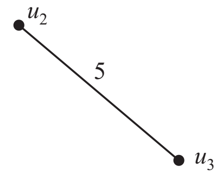

**Fig. 2.13**

## *Solution*:

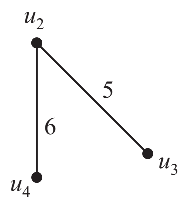
We collect lengths of edges into a table.

| *Edge* | *Cost* |
|---|---|
| $v_2 - v_3$ | 5 |
| $v_2 - v_4$ | 6 |
| $v_4 - v_3$ | 10 |
| $v_2 - v_6$ | 11 |
| $v_4 - v_6$ | 14 |
| $v_2 - v_4$ | 16 |
| $v_4 - v_5$ | 18 |
| $v_5 - v_1$ | 19 |
| $v_1 - v_6$ | 21 |
| $v_5 - v_6$ | 23 |

(*i*) Choose the edge $v_2 - v_3$ (Min. wt.)

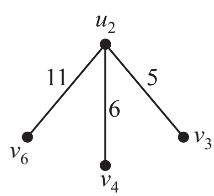



(*ii*) Add the next edge with min. wt.



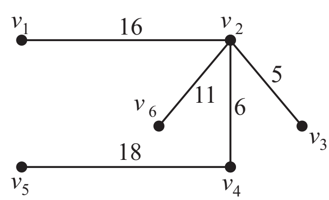

(*iii*) Reject the edge $v_4 - v_3$, because it forms a cycle.

(*iv*) Add the edge $v_2 - v_6$




(*v*) Reject $v_4 - v_6$ since it forms cycle.

(*vi*) Add the edges $v_2 - v_1$, $v_4 - v_5$




Now all the vertices of $G$ are covered, therefore we stop the algorithm.

Cost or weight of Minimal spanning tree

$$= 5 + 6 + 11 + 16 + 18$$

$$= 56.$$

■

■ **Example 2.13:** *Determine a railway network of minimal cost for the cities as shown in Fig. 2.14. Also find minimal cost.*

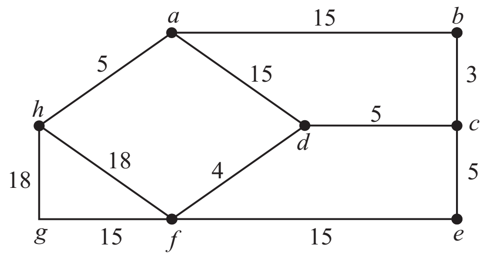



**Fig. 2.14**

***Solution*:**

Collecting lengths of edges into a table, we set:

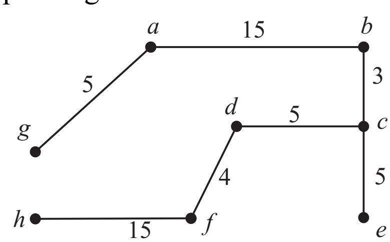
| *Edge* | *Cost* |
|---|---|
| $b-c$ | 3 |
| $d-f$ | 4 |
| $a-g$ | 5 |
| $c-d$ | 5 |
| $c-e$ | 5 |
| $a-b$ | 15 |
| $a-d$ | 15 |
| $f-h$ | 15 |
| $g-h$ | 15 |
| $e-f$ | 15 |
| $f-g$ | 18 |

(*i*) Choose the edges $b-c$, $d-f$, $a-g$, $c-d$ and $c-e$.

(*ii*) We have options: we may choose only one of $a-b$ and $a-d$ for the selection of both creates a circuit. Suppose we choose $a-b$.

(*iii*) Likewise we may choose only one of $g-b$ and $f-h$. Suppose we choose $f-h$.

(*iv*) We then have a spanning tree as illustrated in the Fig. below.



The minimal cost for construction of this tree is

$$= 3 + 4 + 5 + 5 + 5 + 15 + 15$$
$$= 5.2$$ ■

### 2.10.2 Prim's Algorithm

*Input:* A connected weighted graph $G$ with $n$ vertices.

*Output:* A minimal spanning tree $T$.

*Step 1:* Select an arbitrary vertex $v_1$ and an edge $e_1$ with minimum weight incident with vertex $v_1$. This forms initial MST, $T$.

*Step 2:* If edges $e_1$, $e_2$, $e_3$,…,$e_i$ have been chosen involving end points $v_1$, $v_2$,…,$v_{i+1}$. Choose and edge $e_{i+1} = v_j\ v_k$ with $v_j \in T$ and $v_k \in T$ s.f. $e_{i+1}$ has smallest weight among the edges of $G$ with precisely one and in $\{v_i, v_2, \ldots, v_{i+1}\}$.

*Step 3:* Stop after $n - 1$ edges have been chosen. Otherwise go to step 2.

■ **Example 2.14:** *Find the minimal spanning tree of the weighted graph of Fig. 2.15, using Prim's Algorithm.*

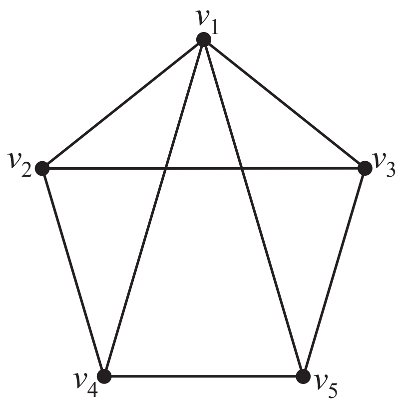
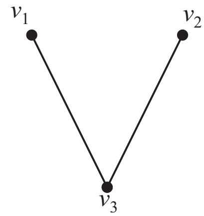



**Fig. 2.15**

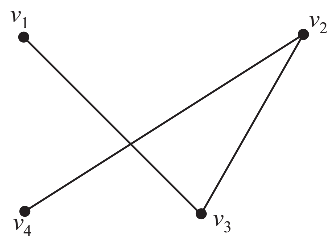
***Solution***:

(*i*) We choose the vertex $v_1$. Now edge with smallest weight incident on $v_1$, is ($v_1$, $v_3$), so we choose the edge on ($v_1$, $v_3$).

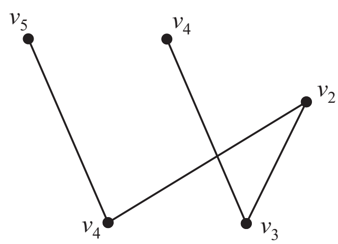
(*ii*) $w(v_1, v_2) = 4$, $w(v_1, v_4) = 3$, $w(v_1, v_5) = 4$, $w(v_1, v_2) = 2$ and $w(v_3, v_4) = 3$. So, we choose the edge $(v_3, v_2)$ since it is of minimum weight.




(*iii*) $w(v_1, v_5) = 3$, $w(v_2, v_4) = 1$. and $w(v_3, v_4) = 3$. We choose the edge $(v_2, v_4)$.




(*iv*) We choose the edge $(v_4, v_5)$. Now all vertices are covered. The minimum spanning tree is produced.




■

■ **Example 2.15:** *Find the minimal spanning tree of the weighted graph using Prim's Algorithm.*

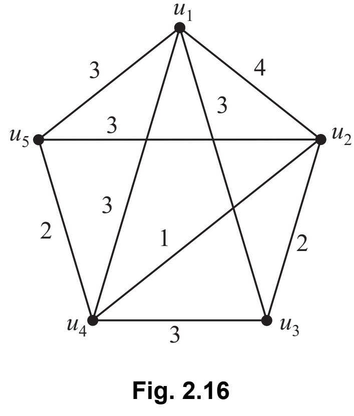


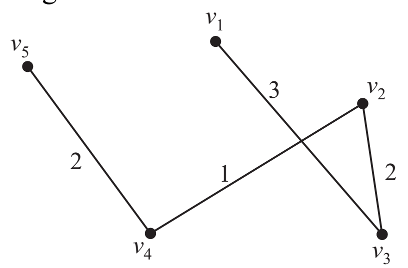

**Fig. 2.16**

***Solution*:**

According to step 1, choose vertex $v_1$. Now edge with smallest weigh incident on $v_1$, is $e = v_1 v_3$ or $v_1 v_5$, choose $e = v_1 v_3$.

Similarly choose the edges $v_3 v_2$, $v_2 v_4$, $v_4 v_5$

The minimal spanning tree is




### 2.10.3 Dijkstra's Algorithm

Dijkstra's Algorithm [(Dijkstra, 1959) and Whiting-Hiller (1960) solved the problem of the shortest path from one location to the other. How can we find the shortest routes  from our home to every place in the town? This problem requires finding the shortest paths from one vertex to all other vertices in the weighted graph as the form of spanning tree.

To find a shortest path from vertex A to vertex E(say) in a weighted graph, we should carry out the following steps:

Step 1: Assign to A the label (–, 0).

Step 2 : Upto E is labeled or no further labels to be assigned, follows the sub steps as:

(*i*) For each labeled vertex $u$ $(x, d)$ and for each unlabeled vertex v adjacent to $u$, compute $d + w(e)$, where $e = uv$.

(*ii*) For each labeled vertex $u$ and adjacent unlableled vertex $v$ giving minimum $d' = d + w(e)$, assign to $v$ the label $(u, d')$.

It a vertex could be labeled $(x, d')$ for various vertices x, we can make any choice.

### *Improved Dijkstra's Algorithm*

To find the length of a shortest path from vertex a to vertex E in a weighted graph, we should proceed as the following:

Step 1: Set $v_1$ = A and assign to this vertex to the permanent label O. (assign very other vertex a temporary label of as where as is a symbol deemed to be larger than any real number)

Step 2 : Until E has been assigned a permanent label (no temporary labels are changed) in (*i*) or (*ii*), follow as:

($i$) Take the vertex $v_i$, most recently acquired a permanent label (say $d$), For each vertex $v$ adjacent to $vi$ has not yet received a permanent label, if $d + w(v_i, v) < t$, the current temporary label of $v$ to $d + w(v_i, v)$.

($ii$) Take a vertex $v$, has a temporary smallest among all temporary labels in the graph. Set $v_i + 1 = v$ and make its temporary label to permanent. If there are so many vertices $v$ which tie for smallest temporary label we can make any choice in particular.

**CTM: Dijkstra's Algorithm – Distance from one Vertex**

Input : A graph with non-negative edge weights and a starting vertex $u$. The weight of edges $xy$ is $w(x, y)$; let $w(x, y) = \infty$, if $xy$ is not an edge.

Idea : Mantain the set $s$ of vertices to which a shortest path from u is known, enlarging $s$ to include all vertices. Maintain a tentative distance $t(z)$ from u to each $z \notin s$, being the shortest length of the shortest $v$, $z$-path yet found.

Initialization : Set $S = \{u\}$; $t(u) = 0$; $t(z) = w(uz)$ for all $z \neq u$.

Iteration : Select a vertex v outside S

such that $t(v) = \min._{z \notin S} t(z)$

Add $v$ to $S$.

Explore edges from $v$ to update tentative distances: for each edge $vz$ with $z \notin S$, update $t(z)$ to min $\{t(z). t(v) + w(vz)\}$.

Note : The iteration continues until $S = V(G)$ or until $t(z) = \infty \ \forall \ z \notin S$.

At the end, set $d(u, v) = t(v) \ \forall \ v$.

## 2.10.4 The Floyd-Warshall Algorithm

This algorithm can be used to find the length of a shortest path between all pairs of vertices in a weighted connected simple graph. However, this alogrithm cannot be used to construct shortest paths. We assign an infinite weight to any pair of vertices not connected by an edge in the graph.

The Floyd-Warshall algorithm is too efficient from the point of view of storage since it can be implemented by just updating the matrix of distances with each change in $k$ (see in algorithm), there is no storage in matrices. It is more faster than Dijkstra's algorithm in many ways.

To find the shortest distance between all pairs of vertices in a weighted graph where the vertices are $v_1, v_2, v_3, \ldots, v_n$ we carry out the following steps.

Step 1 : For i = 1 to n.

Set $d(i, i) = 0 \ \forall \ i \neq j$ iff $v_i v_j$ is an edge.

Let $d(i, j)$ be the weight of this edge.

Otherwise, set $d(i, j) = \infty$.

Step 2 : For $k = 1$ to $n$.

for $i, j = 1$ to $n$, Let $(i, j) = \min \{d(i, j), d(i, k) + d(k, j)]$ the final value of $d(i, j)$ is the shortest distance form $v_i$ to $v_j$.

## 2.11 Directed Trees

Two vertices $u$ and $v$ of a directed graph $G$ are called quasi – strongly connected if there is a vertex $w$ from which there is a direct path to $u$ and $a$ directed path to $v$. If there is a directed path $P$ from $u$ to $v$ then certainly $u$ and $v$ are quasi – strongly connected, because we can take $w$ to be $u$ itself, and then there is a trivial path with no edges from $u$ to $v$ and the path $P$ from $u$ to $v$.

The graph $G$ is said to be quasi – strongly connected if each pair of vertices of $G$ is quasi – strongly connected. It is clear that if a directed graph $G$ is quasi – strongly connected, then the underlying non directed graphs will be connected. The digraph (Fig. 2.17) Quasi – Strongly connected.

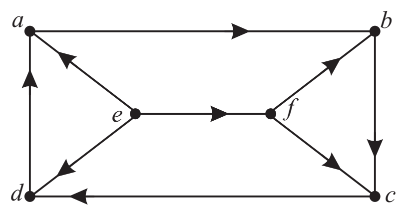



**Fig. 2.17: Quasi-strongly Connected Graph**

*A digraph a has a directed spanning tree iff* G *is quasi – strongly connected.*

A directed forest is a collection of directed trees. The height of vertex $v$ in a directed forest in the length of the longest directed path from $v$ to $a$ leaf. The height of a nonempty tree is the height of its root.

The level of a vertex $v$ in a forest is the length of the path to $v$ from the root of the tree to which it belongs. A directed tree $T$ is said to have degree $k$ if $k$ is the maximum of the out-degrees of all the vertices in $T$.

### *Theorem 2.10*

Let $G$ be a digraph. Then the following are equivalent:

(*i*) *G is quasi – strongly connected.*

(*ii*) *There is a vertex r in G such that there is a directed path from r to all the remaining vertices of G.*

***Proof***:

It is clear that $(ii) \Rightarrow (i)$

On the other hand, Let $G$ is quasi – strongly connected and we consider its vertices $v_1, v_2, ..., v_n$. There is a vertex $w_2$ from which there is a path to $v_1$ and a path to $v_2$. There might also be a vertex $w_3$ from which there is a path to $w_2$ and $w_3$, and so on finally we conclude that there is a vertex $w_n$ from which there is a path to $w_{n-1}$ and a path to $v_n$.

It is clear that, there is a directed path from $w_n$ to each vertex $v_i$ of $G$ since $w_n$ is connected to $v_1, \dots, v_{n-1}$ through $w_{n-1}$. Hence $(i) \Rightarrow (ii)$. ■

## 2.12 Solved Examples

■ **Example 2.16:** *Suppose that a tree $T$ has $N_1$ vertices of degree 1. $N_2$ vertices of degree 2, $N_3$ vertices of degree 3,..., $N_k$ vertices of degree k. Prove that $N_1 = Q_1 + N_3 + 2N_4 + 3N_5 +, ..., + (k-2)\, N_k$.*

***Solution***:

We consider a tree $T$.

The total number of vertices $= N_1 + N_2 + N_3 +, ..., + N_k$

Sum of degrees of vertices $= N_1 + 2N_2 + 3N_3 +, ..., + N_k$.

$\therefore$ Total number of edges in $T = N_1 + N_2 +, ..., + N_k - 1$.

by handshaking property, we have

$$N_1 + 2N_2 + 3N_3 +, ..., + kN_k = 2(N_1 + N_2 +, ..., + N_k - 1)$$

Solving, we get

$$N_3 + 2N_4 + 3N_5 +, ..., + (k-2)\, N_k = N_1 - 2$$

$$\Rightarrow \quad N_1 = 2 + N_3 + 2N_4 + 3N_5 +, ..., + (k-2)\, N_k.$$ ■

■ **Example 2.17:** *Which trees are complete bipartite graphs?*

***Solution***:

Let $T$ be a tree, which is a complete bipartite graph.

Let $T = K_{m,n}$ then the number of vertices in $T = m + n$

$\therefore$ The tree contains $(m + n - 1)$ number of edges.

But the graph $k_{m,n}$ number of edges.

Therefore, $\quad m + n - 1 = mn$.

$\Rightarrow \quad mn - m - n + 1 = 0$

$\Rightarrow \quad m(n-1) - 1(n-1) = 0$

$\Rightarrow \quad (m-1)(n-1) = 0$

$\Rightarrow \quad m = 1$ or $n = 1$.

This means that $T$ is either $k_{1,n}$ or $k_{m,1}$.

*i.e.* $T$ is a star. ■

■ **Example 2.18:** *Let $G = (V, E)$ be a loop free undirected graph with $|V| = n$. Prove that G is a tree if and only if, $P(G, \lambda) = \lambda(\lambda - 1)^{n-1}$.*

### *Solution:*

If $G$ is a tree. We consider $G$ is a rooted tree. There are $\lambda$ choices for closing the root of $G$ and $(\lambda - 1)$ choices for colouring each of its descendents. The result followed by rule of product conversely, if $P(G, \lambda) = \lambda(\lambda - 1)^{n-1}$, then since the factor $\lambda$ occurs only once, the graph $G$ is connected. $P(G, \lambda) = \lambda(\lambda - 1)^{n-1}$, then since the factor $\lambda$ occurs only once, the graph $G$ is connected.

$$P(G, \lambda) = \lambda(\lambda - 1)^{n-1} = \lambda^n - (n-1)\,\lambda^{n-1} +,\ldots,+ (\lambda)^{n-1}\lambda.$$

$\Rightarrow$ $G$ has $n$ vertices and $(n - 1)$ edges.

Hence $G$ is a tree. ■

■ **Example 2.19:** *Using the Kruskal's Algorithm, find a minimal spanning tree for the weighted graph given below (Fig. 2.18).*

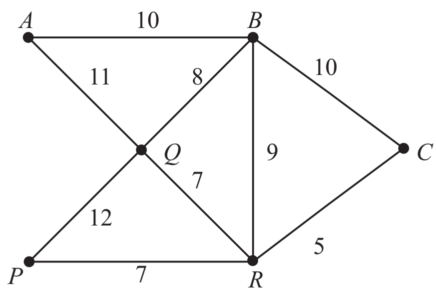


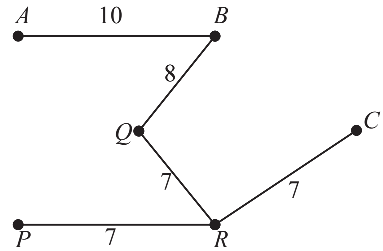

**Fig. 2.18**

### *Solution:*

The given graph has 6 vertices hence a spanning tree will have $(6 - 1) = 5$ edges.

| *Edge* | *Weight* | *Select* |
|---|---|---|
| $C - R$ | 5 | ✓ |
| $P - R$ | 7 | ✓ |
| $Q - R$ | 7 | ✓ |
| $B - Q$ | 8 | ✓ |
| $B - R$ | 9 | × |
| $A - B$ | 10 | ✓ |
| $B - C$ | 10 | × |
| $A - R$ | 11 | × |
| $P - Q$ | 12 | × |

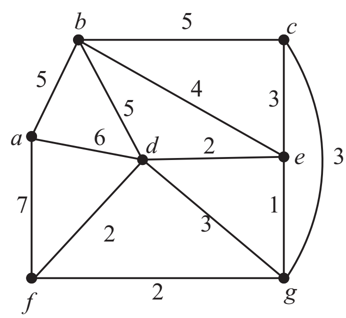
A minimal spanning tree of given graph contains the edges $C-R$, $P-R$, $Q-R$, $B-Q$, $A-B$. The tree will be




The weight of the tree will be

$$5 + 7 + 7 + 8 + 10 = 37 \text{ units.}$$ ■

■ **Example 2.20:** *Use Prim's algorithm find an optimal graph for the graph in Fig. 2.19.*




## ***Solution***:

An optimal tree can be generated as follows:

*To initialize:* $i = 1, P = \{a\}, N = \{b, c, d, e, f, g\}, T = \phi.$

*I iteration:* $T = [\{a, b\}], P = \{a, b\}, N = \{c, d, e, f, g\}, i = 2.$

*II iteration:* $T = [\{a, b\}, \{b, e\}], P = \{a, b, e\}, N = \{c, d, f, g\},$ $i = 3.$

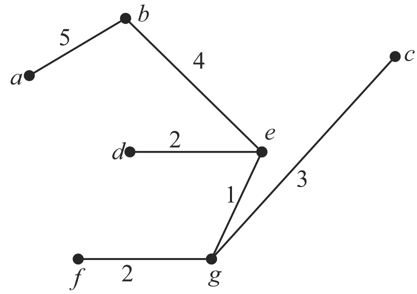
*III iteration:* $T = [\{a, b\}, \{b, e\}, \{e, g\}], P = \{a, b, e, g\},$
$N = \{c, d, f\}, i = 4$

*IV iteration:* $T = [\{a, b\}, \{b, e\}, \{e, g\}, \{d, e\}, \{f, g\}]$

*V iteration:* $T = [\{a, b\}, \{b, e\}, \{e, g\}, \{d, e\}, \{f, g\}]$
$P = \{a, b, e, g, d, f\}, N = \{c\}, i = 6.$

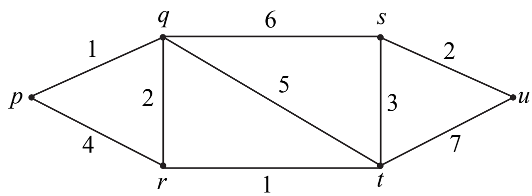
*VI iteration:* $T = [\{a, b\}, \{b, e\}, \{e, g\}, \{d, e\}, \{f, g\}, \{c, g\}]$
$P = \{a, b, e, g, d, f, c\} = V, n = \phi, i = 7 = |V|$

Hence it can be proved that $T$ is an optimal spanning tree of weight 17 for $G$.




of $T = 1 + 2 + 2 + 3 + 4 + 5 = 17$. ■

■ **Example 2.21:** *Find the shortest path to the graph given below from p to u by using Dijkstra's algorithm.*




***Solution***:

We can arrange the initial labelling by:

| Vertex (V) | $p$ | $q$ | $r$ | $s$ | $t$ | $u$ |
|---|---|---|---|---|---|---|
| $L(v)$ | o | α | α | α | α | α |
| $T$ | $\{p$ | $q$ | $r$ | $s$ | $t$ | $u\}$ |

*Iteration 1:* $U = p$, has $L(u) = 0$

T becomes $T - \{p\}$

Then, two edges incident $p$ i.e. $pq$ and $pr$

where $q$ and $r \in T$.

$L(q) = \min\ \{\text{old } L(q), L(p) + w(pq)\}$

$= \min\ \{a, 0 + 1.0\} = 1.0.$

$L(r) = \min.\ \{\text{old } L(r), L(p) + w(pr)\}$

$= \min.\ \{\alpha, 0 + 4.0\} = 4.0$

Hence minimum label is $L(q) = 1.0$

| Vertex (V) | $p$ | $q$ | $r$ | $s$ | $t$ | $u$ |
|---|---|---|---|---|---|---|
| $L(v)$ | o | 1.0 | 4.0 | α | α | α |
| $T$ | $\{p$ | $q$ | $r$ | $s$ | $t$ | $u\}$ |

*Iteration 2:* $U = q$, the permanent label of $q$ is 1.0 $T$ becomes $T - \{q\}$. There are three edges incident with $q$ i.e. $qr$, $qs$ and $qt$ where $r, s, t \in T$

$L(r) = \min \{\text{old } L(r), L(q) + w(qr)\}$
$= \min \{4.0, 1.0 + 2.0\} = 3.0.$

$L(s) = \min. \{\text{old } L(s), L(q) + w(qs)\}$
$= \min. \{\alpha, 1.0 + 6.0\} = 7.0$

$L(t) = \min. \{\text{old } L(t), L(q) + w(qt)\}$
$= \min. \{\alpha, 1.0 + 5.0\} = 6.0$

| Vertex (V) | $p$ | $q$ | $r$ | $s$ | $t$ | $u$ |
|---|---|---|---|---|---|---|
| $L(v)$ | 0 | 1.0 | 3.0 | 7.0 | 6.0 | $\alpha$ |
| $T$ | { | | $r$ | $s$ | $t$ | $u$} |

Hence minimum label is $L(r) = 3.0$

*Iteration 3:* $U = r$, the permanent label of $t$ is 3.0
$T$ becomes $T - \{r\}$.
There is only one edges incident with $r$
i.e. $r, t$ where $t \in T$

$L(r) = \min \{\text{old } L(r), L(r) + w(rt)\}$
$= \min \{6.0, 3.0 + 1.0\} = 4.0.$

| Vertex (V) | $p$ | $q$ | $r$ | $s$ | $t$ | $u$ |
|---|---|---|---|---|---|---|
| $L(v)$ | 0 | 1.0 | 3.0 | 7.0 | 4.0 | $\alpha$ |
| $T$ | { | | | $s$ | $t$ | $u$} |

Hence minimum label is $L(r) = 4.0$

*Iteration 4:* $U = t$, the permanent label of $t$ is 4.0
$\therefore$ $T$ becomes $T - \{t\}$.
There are two edges incident with $t$
i.e. $ts$ and $tu$ where $s, u \in T$

$L(s) = \min \{\text{old } L(s), L(t) + w(tu)\}$
$= \min \{7.0, 4.0 + 3.0\} = 7.0.$

$L(u) = \min \{\text{old } L(u), L(t) + w(tu)\}$
$= \min \{\alpha, 4.0 + 7.0\} = 11.0.$

| Vertex (V) | $p$ | $q$ | $r$ | $s$ | $t$ | $u$ |
|---|---|---|---|---|---|---|
| $L(v)$ | 0 | 1.0 | 3.0 | 7.0 | 4.0 | 11.0 |
| $T$ | { | | | $s$ | | $u$} |

Hence minimum label is $L(s) = 7.0$

*Iteration 5:* $U = s$, the permanent label of $s$ is 7.0

$\therefore$ $T$ becomes $T - \{s\}$.

There is only one edges incidents on $s$

i.e. $su$ where $u \in T$

$L(u) = \min \{\text{old } L(u), L(s) + w(su)\}$

$= \min \{11.0,\ 7.0 + 2.0\} = 9.0.$

| Vertex (V) | $p$ | $q$ | $r$ | $s$ | $t$ | $u$ |
|---|---|---|---|---|---|---|
| $L(v)$ | 0 | 1.0 | 3.0 | 7.0 | 4.0 | 9.0 |
| $T$ | { | | | | | $u$} |

Hence the minimum label is $L(u) = 9.0$

Since $U = u$ is the only choice. Hence the shortest distance between $p$ to $u$ is $q$ and the shortest path is $\{p, q, r, t, s, u\}$ ■

## SUMMARY

1. A **tree** is a connected acyclic graph.
2. A connected graph is a tree if and only if a unique, simple path runs between any two vertices.
3. A connected graph with $n$ vertices is a tree if and only if it has exactly $n - 1$ edges.
4. A **spanning tree** of a connected graph contains every vertex of the graph.
5. Every connected graph has a spanning tree.
6. **Kruskal's algorithm**, the **DFS method**, and **the BFS method** and find spanning trees.
7. A **minimal spanning tree** of a connected weighted graph weighs the least.
8. **Kruskal's algorithm** and **Prim's algorithm** can find minimal spanning trees.
9. A specially designated vertex in a tree is the root of the tree. A tree with a root is **rooted tree**.
10. The **subtree** rooted at $v$ consists of $v$, its descendants, and the edges incident with them.
11. The **level** of a vertex is the length of the path from the root to the vertex.
12. The **height** of a tree is the maximum level of any leaf in the tree.
13. In an **ordered rooted tree** the children of every vertex are ordered.
14. An $m$- ary ***tree*** is a rooted tree in which every vertex has at most $m$ children. It is **binary** if $m = 2$, and **ternary** if $m = 3$.
15. An $m$-ary tree is **full** if every internal vertex has $m$ children.
16. An $m$-ary tree is **balanced** if all leaves fall on the same level or two adjacent levels.

**17.** An $m$-ary tree is **complete** if all leaves lie at the same level.

**18.** An $m$-ary tree of height $h$ has at most $m^h$ leaves.

**19.** For an $m$-ary tree of height $h$ with $l$ leaves, $h \geq [\log_m l]$.

**20.** If it is full and balanced, $h = [\log_m l]$.

**21.** **Preorder, inorder,** and **postorder traversal** are three ways to visit every vertex of a binary tree.

**22.** An algebraic expression with only binary operators can be written in **prefix, infix**, or **postfix form**. In prefix form, each operator precedes its operands. The other two forms behave similarly.

**23.** An algebraic expression containing only binary operators can be represented by a **binary expression tree**.

**24.** A **binary search tree** is homogeneous with every element in the left subtree of every vertex $v$ less than $v$ and every right subtree element is greater than $v$.

**25.** A **Huffman code,** a variable-length code, minimize the length of encoded messages.

**26.** A **decision tree** is an $m$-ary tree in which a decision is made at each internal vertex.

## EXERCISES

1. Show that a Hamiltonian path is a spanning tree.
2. Let $G$ be a graph with $k$ components, where each component is a tree. If $n$ is the number of vertices and $m$ is the number of edges in $G$. Prove that $n = m + k$.
3. Show that the complete bipartite graph $K_{r,s}$ is not a tree if $r \geq 2$.
4. Show that the number of vertices in a binary tree is always odd.
5. Prove that the maximum number of vertices in a binary tree of depth $d$ is $2^d - 1$ when $d \geq 1$.
6. Let $T = (V, E)$ be a complete $m$–ary tree of height $h$ with l leaves. Then prove that $l \leq m^h$ and $h \geq (\log m^l)$.
7. If $G = (V, E)$ is an undirected graph then $G$ is connected iff $G$ has a spanning tree.
8. If every tree $T = (V, E)$ then prove that $|V| = |E| + 1$.
9. If $G = (V, E)$ be a loop–free undirected graph, and $\deg(v) \geq 2$ $\forall$ $v \in V$, prove that $G$ contains a cycle.
10. Explain that, a graph with $n$ vertices, $n - 1$ edges, and no circuit is connected.

## Suggested Readings

1. **A.V.Aho**, *et al*., *Data Structures and Algorithmis,* Addison-Wesley, Reading, MA, 1983, pp. 75-196, 230-292.
2. **T.H.Carmen** *et al*., *Introduction to Algorithms,* McGraw-Hill, New York, 1990.
3. **D.I.A. Cohen**, *Basic Techniques of Combinatorial Theory,* Wiley, New York, 1978, pp. 228-282.
4. **R.L.Kruse**, *Data Structure and Program Design,* 2nd ed., Prentice Hall, Englewood Cliffs, NJ, 1987, pp. 318-498, 512-517.
5. **J.A.McHugh**, *Algorithmic Graph Theory,* Prentice-Hall, Englewood Cliffs, NJ, 1990.
6. **S. Sahni**, *Concepts in Discrete Mathematics,* 2nd ed., Camelot, Fridley, MN, 1985, pp. 391-448.
7. **S.S. Skieno**, *Implementing Discrete Mathematics*, Addison-Wesley, Reading, MA, 1990.
8. **S.K. Yadav**, *Elements of Graph Theory*, Ane Books New Delhi, 2011.

❑❑❑

# 3 Planar Graphs

**Kazimierz Kuratowski (1896–1980)**

Kazimierz Kuratowski (1896-1980) was a Polish mathematician born in Warsaw. His father was a lawyer. He studied engineering in the university of Glasgow, Scotland and completed it in 1913. Due to First World War in 1914, he could not returns to Glasgow. In 1915, Kuratowski studied mathematics at the university of Warsaw under the guidance of the reputed logician Jan Lukasiewicz and graduated in 1919. He received his Ph.D. in 1921.
Kuratowski became professor of mathematics at Lvov Technical University in 1927. Six years later, he returned to the university of Warsaw and hold academic and administrative both posts where he worked untill 1966. He was editor-in-chief of several research journals and council member of various research and apex bodies of that time.
Kuratowski wrote numerous articles for professional journals and contributed to topology, analysis and graph theory.

## 3.1 Introduction

In Graph Theory, a planar graph is a graph which can be embedded in the plane *i.e.*, it can be drawn on the plane in such a manner that its edges intersect only at their endpoints. A planar graph which has already drawn in the plane without edge intersection is called a plane graph or planar embedding of the graph. A plane graph can be defined as a planar graph with a mapping from every node to a point in $2D$ space, and from every edge to a pane curve, such that the extreme points of each curve are the points mapped from its end nodes, and all the curves are disjoint except on their extreme points. Plane graphs can be encoded by combinatorial maps.

It can easily be seen that a graph that can be drawn on the plane can be drawn on the sphere as well and vice-a-versa.

The equivalence class of topologically equivalent drawing on the sphere is called a planar map. Although a planar graph has an external or unbounded face, none of the faces of a planar map have a particular status.


S. K. Yadav, *Advanced Graph Theory*,
https://doi.org/10.1007/978-3-031-22562-8_3

A generalization of planar graphs are graphs which can be drawn on a surface of a given genus. In this terminology, planar graphs have graph genus *O*, since the plane and the sphere are surfaces of genus *O*. The following are the planar graphs. (Fig. 3.1).

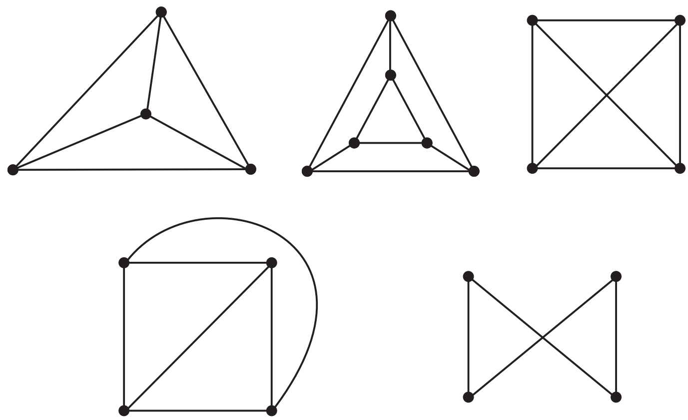


**Fig. 3.1: Planar graphs**

## 3.2 Geometrical Representation of Graphs

As we have defined a graph $G$ in chapter 1 as $G = (V, E, \phi)$ where the set $V$ consists of the five objects named $a$, $b$, $c$, $d$ and $e$ *i.e.* $V = \{a, b, c, d, e\}$ and the set $E$ consists of seven objects (none of which is in set $V$) named 1, 2, 3, 4, 5, 6, and 7, *i.e.*

$$E = \{1, 2, 3, 4, 5, 6, 7\}$$

and the relationship between the two sets is defined by the mapping, $\Psi$, which consists of combinatorial representation of the graph.

$$\psi = \begin{cases} 1 & \to (a, c) \\ 2 & \to (c, d) \\ 3 & \to (a, d) \\ 4 & \to (a, b) \\ 5 & \to (b, d) \\ 6 & \to (d, e) \\ 7 & \to (b, e) \end{cases}$$

Where, the symbol $1 \to (a, c)$ represents that object 1 from set $E$ is mapped onto the pair $(a, c)$ of objects from set $V$.

It can be represented by means of the Fig. 3.2. It truly shows that the graph can be represented by means of such configuration.

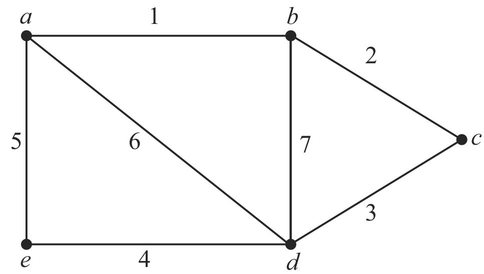



**Fig. 3.2: Geometrical Representation of Graph**

### *Theorem 3.1*

*The complete graph $K_5$ is non-planar.*

### *Proof:*

We attempt to draw $K_5$ in the plane. We first start with a pentagon: As shown in Fig. 3.3

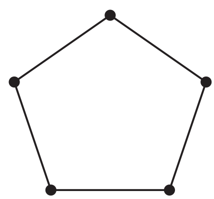


**Fig. 3.3 (a)**

A complete graph contains an edge between every pair of vertices, so there is an edge between *a* and *c*. This may as well be inside the pentagon (as if it is outside then we just adjust the following argument appropriately).

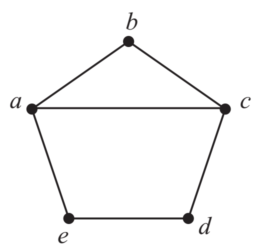



**Fig. 3.3 (b)**

Now we add the edge between *b* and *e* (this must be outside the pentagon as it cannot cross $\{a, c\}$, the edge between *a* and *b* (inside so as to not cross $\{b, e\}$), and then between *c* and *e* (outside so as to not cross $\{a, d\}$ [As Fig. 3.3(c)]

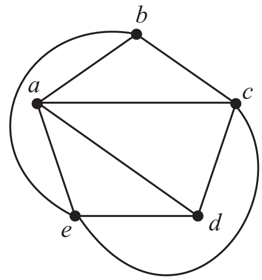



**Fig. 3.3 (c)**

All these edges were forces into position and we have no choice. It remains to add an edge between $b$ and $d$. We cannot add it inside (since it would cross $\{a, c\}$ nor can we add it outside (since it would cross $\{c, e\}$).

Consequently $K_5$ is non-planar. ■

## 3.3 Bipertite Graph

A simple graph $G = (V, E)$ is called bipertite if $V = V_1 \cup V_2$ with $V_1 \cap V_2 = \phi$ and every edge of $G$ is of the form $\{a, b\}$ with one of the vertices $a$ and $b$ in $V_1$ and the other in $V_2$.

If every vertex in $V_1$ is joined to every vertex in $V_2$ we can obtain a complete bipertite graph. We may write $K_{m,\,n}$ for the complete bipertite graph with $|V_1| = m$ and $|V_2| = n$.

Here $|E| = mn$. (see examples in fig 3.4)

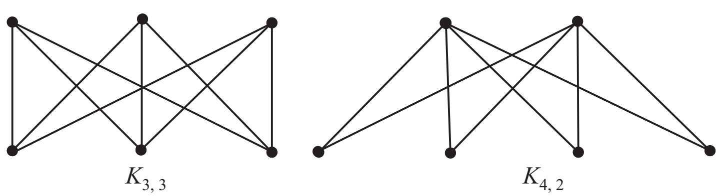



**Fig. 3.4: Complete Bipertite Graphs**

### *Theorem 3.2*

*The complete bipertite graph $K_{3,\,3}$ is nonplanar.*

### *Proof*:

Let $V_1 = \{a, b, c\}$ and $V_2 = \{x, y, z\}$ and draw a hexagonal circuit:

$a \to x \to b \to y \to c \to z \to a$

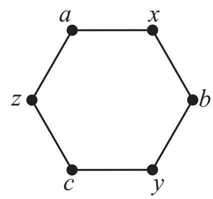



**Fig. 3.5**

The proof is completed by observing that two of the edges $\{a, y\}$, $\{b, z\}$ or $\{c, x\}$ must both lie inside or both outside the hexagon and hence must cross.

■

## 3.4 Homeomorphic Graph

Two graphs $G_1$ and $G_2$ such that $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be homeomorphic if and only if $G_2$ can be obtained from $G_1$ by the insertion or deletion of a number of vertices of degree two (deg. 2).

The following three graphs are homeomorphic (Fig. 3.6)

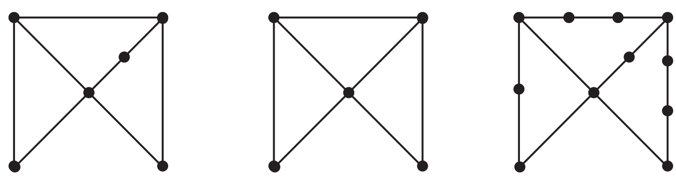


**Fig. 3.6 Homeomorphic Graph**

One can think of homeomorphic graphs as being as the same shape. Adding or deleting a vertex of degree two does not change the shape of the edges but simply replaces a single edge by a pair of edges taking the same shape (or vice versa).

■ **Example 3.1:** *If graph $(p_1, q_1)$ and $(p_2, q_2)$ del are homeomorphic then $p_1 + q_2 = p_2 + q_1$.*

***Solution***:

Let $(p_1, q_1) = G_1$, and $(p_2, q_2) = G_2$ he homeomorphic graphs. Therefore $G_1$ and $G_2$ can be from $a$ $(p, q)$ graph $G$ by a series of elementary subdivisions respectively (say $r$ and $s$).

In each elementary subdivision, the number of points as well as the number of edges increase by one.

Hence $\quad p_1 = p + r, q_1 = q + r, p_2 = p + s, q_2 = q + s$

Hence $\quad p_1 + q_2 = b + r + q + s = (p + s) + (q + r) = p_2 + q_1$

(It shows necessary and sufficient condition for a graph to be planar). ■

## 3.5 Kuratowski's Graphs

The graphs $K_5$ and $K_{3,3}$ are called Kuratowski's graphs. A graph is planar if and only if it has no subgraph, homeomorphic to $K_5$ or $K_{3,3}$.

Since planarity is such a fundamental property, it is clearly of importance to know which graphs are planar and which are not.. We have already noted that, in particular $K_5$ and $K_{3,3}$ are non-planar and that any proper subgraph of either of these graphs is planar. A remarkable simple characterisation of planar graphs was given by Kuratowski (1930). This can be proved by two lemmas.

***Lemma-3.1:*** If $G$ is non-planar, then every subdivision of $G$ is non-planar.

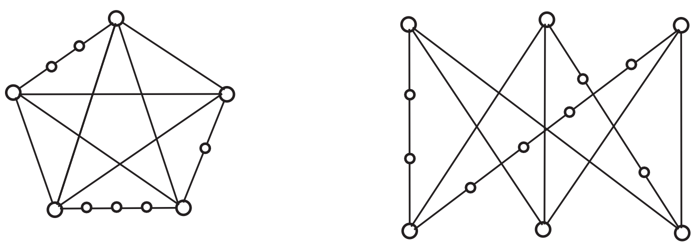
***Lemma-3.2:*** If $G$ is planar, they every subgraph of $G$ is planar.

### *Proof*:

Since $K_5$, and $K_{3,3}$ are nonplanar, we can see that if $G$ is planar, then $G$ cannot contain a subdivision of $K_5$ or of $K_{3,3,}$ (Fig. 3.7). Kuratowski showed that this necessary condition is also sufficient.

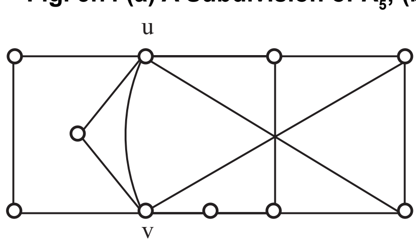
Before proving Kuratowski's theorem, we need to establish two more simple lemmas.

Let $G$ be a graph with a 2-vertex cut $\{u, v\}$. Then there exist edge-disjoint subgraph $G_1$ and $G_2$ such that $V(G_1) \cap V(G_2) = \{u, v\}$ and $G_1 \cup G_2 = G$. Consider such a separation of $G$ into subgraphs. In both $G_1$ and $G_2$ join $u$.

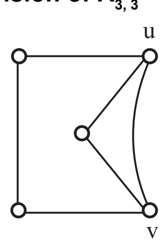


(a) (b)

**Fig. 3.7: (a) A Subdivision of $K_5$; (b) a Subdivision of $K_{3,3}$**




*G:*




*$G_1$*

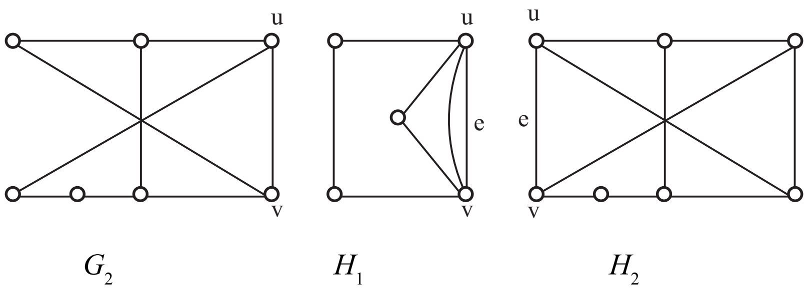



**Fig. 3.8**

and $v$ by a new edge $e$ to obtain graphs $H_1$ and $H_2$, as in Fig. 3.8. Clearly $G = (H_1 \cup H_2) = e$. It can also easily seen that $\varepsilon\,(H_i) < \varepsilon(G)$ for $i = 1, 2$.

■

### *Lemma 3.3*

*If G is non-planar, than at least one of $H_1$ and $H_2$ is also non-planar.*

***Proof*:**

Contradictingly, suppose, both $H_1$ and $H_2$ are planar. Let $H_1$ be a planar embedding of $H_1$, and let $f$ be a face of $\bar{H}_1$ incident with $e$. If $\bar{H}_2$ is an embedding of $H_2$ in $f$ such that $H_1$ and $\bar{H}_2$ have only the vertices $u$ and $v$ and the edge $e$ is common, then $\left(\bar{H}_1 \cup H_2\right) - e$ is a planar embedding of $G$.

This contradiction hypothesis shows that $G$ is nonplanar.

### *Lemma 3.4*

*Let G be a non-planar connected graph that contains no subdivision of $K_5$ or $K_{3,\,3}$ and has as few edges as possible. Then G is simple and 3–connected.*

***Proof*:**

By contradiction. Let $G$ satisfy the hypotheses of the lemma. Then $G$ is clearly a minimal nonplanar graph, and therefore must be a simple block. If $G$ is not 3-connected, let $\{u, v\}$ be a 2-vertex cut of $G$ and let $H_1$ and $H_2$ be the graphs obtained from this cut as described above. By lemma at least one of $H_1$ and $H_2$, say $H_1$, is nonplanar. Since $\varepsilon(H_1) < \varepsilon(G)$, $H_1$ must contain a subgraph $K$ which is a subdivision of $K_5$ or $K_{3,\,3,}$; moreover $K \not\subseteq G$, and so the edge $e$ is in $K$. Let $P$ be a $(u, v)$-path in $H_2 - e$. Then $G$ contains the subgraph $(K \cup P) - e$, which is a subdivision of $K$ and hence a subdivision of $K_5$ or $K_{3,\,3}$. This contradiction establishes the lemma. ■

### *Theorem 3.3*

*A graph is planar if and only if it contains no subdivision of $K_5$ and $K_{3, 3,}$*

### *Proof*:

We have already noted that the necessity follows from lemmas 1 and 2. We shall prove the sufficiency by contradiction.

If possible, choose a nonplanar graph $G$ that contains no subdivision of $K_5$ or $K_{3, 3}$ and has as few edges as possible. From lemma 4 it follows that $G$ is simple and 3–connected. Clearly $G$ must also be a minimal nonplanar graph.

Let $u\ v$ be an edge of $G$, and let $H$ be a planar embedding of the planar graph $G - u\ v$. Since $G$ is 3-connected, $H$ is 2-connected $u$ and $v$ are contained together in a cycle of $H$. Choose a cycle $C$ of $H$ that contains $u$ and $v$ and is such that the number of edges in Int $C$ is as large as possible.

Since $H$ is simple and 2-connected, each bridge of $C$ in $H$ must have at least two vertices of attachment. Now all outer bridges of $C$ must be 2-bridges that overlap $uv$ because, if some outer bridge were a $k$-bridge for $k \geq 3$ or a 2-bridge that avoided $uv$, then there would be a cycle $C'$ containing $u$ and $v$ with more edges in its interior than $C$, contradicting the choice of $C$. These two cases are illustrated in Fig. 3.9 (with $C'$ indicated by heavy lines).

In fact, all outer bridges of $C$ in $H$ must be single edges. For if a 2-bridge with vertices of attachment $x$ and $y$ had a third vertex, the set $\{x, y\}$ would be a 2-vertex cut of $G$, contradicting the fact that $G$ is 3-connected.

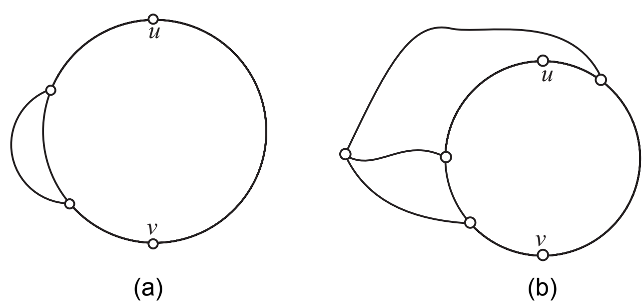



**Fig. 3.9: Vertex Attachment**

Two cases now arise, depending on whether $B$ has a vertex of attachment different from $u$, $v$, $x$ and $y$ or not.

Case 1 $B$ has a vertex of attachment different from $u$, $v$, $x$ and $y$. We can choose the notation so that $B$ has a vertex of attachment $v_1$ in $C(x, u)$ (see Fig. 3.10). We consider two subcases, depending on whether $B$ has a vertex of attachment in $C(y, v)$ or not.

Case 1($a$) $B$ has a vertex of attachment $v_2$ in $C(y, v)$. In this case there is a $(v_1, v_2)$-path $P$ in $B$ that is internally-disjoint from $C$. But then $(C \cup P) + \{uv, xy\}$ is a subdivision of $K_{3, 3}$ in $G$, a contradiction (see Fig. 3.10).

Case 1($b$) $B$ has no vertex of attachment in $C(y, v)$. Since $B$ is skew to $uv$ and to $xy$, $B$ must have vertices of attachment $v_2$ in $C\{u, y\}$ and $v_3$ in $C\{v, x\}$. Thus $B$ has three vertices of attachment $v_1$, $v_2$ and $v_3$. There exists a vertex $v_0$ in $V(B) \setminus V(C)$ and three paths $P_1$, $P_2$ and $P_3$ in $B$ joining $v_0$ to $v_1$, $v_2$ and $v_3$, respectively, such that, for $i \neq j$, $P_i$ and $P_j$ have only the vertex $v_0$ in common. But now $(C \cup P_1 \cup P_2 \cup P_3) + \{uv, xy\}$ contains a subdivision of $K_{3, 3}$ a contradiction. This case is illustrated in Fig. 3.11. The subdivision of $K_{3, 3}$ is indicated by heavy lines.

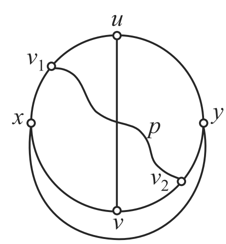



**Fig. 3.10**

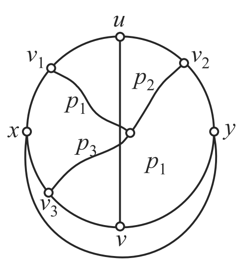



**Fig. 3.11**

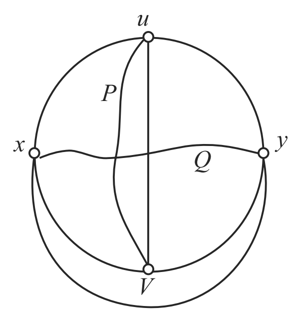



**Fig. 3.12**

Case 2 $B$ has no vertex of attachment other than $u$, $v$, $x$ and $y$. Since $B$ is skew to both $uv$ and $xy$, it follows that $u$, $v$, $x$ and $y$ must all be vertices of attachment of $B$. Therefore there exists a $(u, v)$-path $P$ and an $(x, y)$-path $Q$ in $B$ such that ($i$) $P$ and $Q$ are internally-disjoint from $C$, and ($ii$) $|V(P) \cap V(Q)| \geq 1$. We consider two subcase, depending on whether $P$ and $Q$ have one or more vertices in common.

Case 2a $|V(P) \cap V(Q)| = 1$. In this case $(C \cup P \cup Q) + \{uv, xy\}$ is a subdivision of $K_5$ in $G$, again a contradiction (see Fig. 3.12).

Case 2b $|V(P) \cap V(Q)| \geq 2$. Let $u'$ and $v'$ be the first and last vertices of $P$ on $Q$, and let $P_1$ and $P_2$ denote the $(u, u')$- and $(v', v)$-sections of $P$. Then $(C \cup P_1 \cup P_2 \cup Q) + \{uv, xy\}$ contains a subdivision of $K_{3,3}$ in $G$, once more a contradiction (see Fig. 3.13).

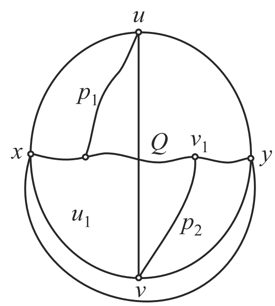



**Fig. 3.13**

Thus all the possible cases lead to contradictions, and the proof is complete ■

## 3.6 Dual Graphs

A plane graph $G$ partition the rest of the plane into a number of connected regions; the closures of these regions are called the faces of $G$. Fig. 3.14 shows a plane graph with six faces $f_1, f_2, f_3, f_4, f_5$ and $f_6$. The notion of a face applies also to embeddings of other surfaces. We denote $F(G)$ and $\phi(G)$ respectively as the set of faces and the number of faces of a plane graph $G$. Each plane graph has exactly one unbounded face, called the exterior face. In the plane graph of Fig. 3.14, $f_1$ in the exterior face.

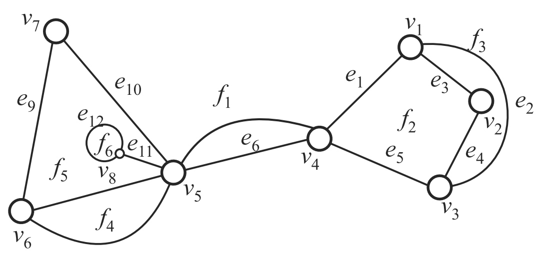



**Fig. 3.14: A Plane Graph with six Faces**

We denote the boundary of a face $f$ of a plane graph $G$ by $b(f)$. If $G$ is connected, then $b(f)$ can be regarded as a closed walk in which each cut edge of $G$ in $b(f)$ is traversed twice; when $b(f)$ contains no cut edges, it is a cycle of $G$. For example, in the plane graph of Fig. 3.14

$$b(f_2) = v_1 e_3 v_2 e_4 v_3 e_5 v_4 e_1 v_1$$

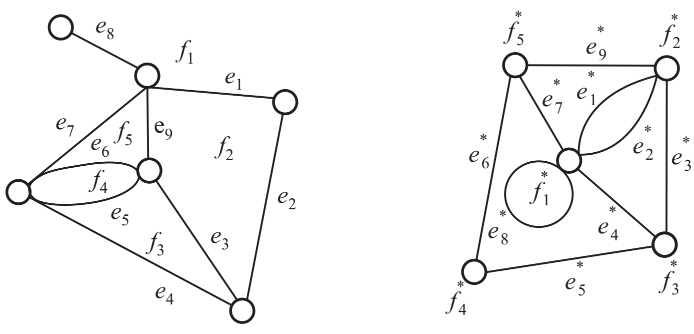
and

$$b(f_5) = v_7 e_{10} v_5 e_{11} v_8 e_{12} v_8 e_{11} v_5 e_8 v_6 e_9 v_7$$

A face $f$ is said to be incident with the vertices and edges in its boundary. If $e$ is a cut edge in a plane graph, just one face is incident with $e$; otherwise, there are two faces incident with $e$. We say that an edge separates the faces incident with it. The degree, $d_G(f)$, of a face $f$ is the number of edges with which it is incident (that is, the number of edges in $b(f)$), cut edges being counted twice. In Fig. 3.14 $f_1$ is incident with the vertices $v_1$, $v_3$, $v_4$, $v_5$, $v_6$, $v_7$ and the edges $e_1$, $e_2$, $e_5$, $e_6$, $e_7$, $e_9$, $e_{10}$; $e_1$ separates $f_1$ from $f_2$ and $e_{11}$ separates $f_5$ from $f_5$; $d(f_2) = 4$ and $d(f_5) = 6$.

Given a plane graph $G$, one can define another graph $G^*$ as follows: corresponding to each face $f$ of $G$ there is a vertex $f^*$ of $G^*$, and corresponding to each edge $e$ of $G$ there is an edge $e^*$ of $G^*$, two vertices $f^*$ and $g^*$ are joined by the edge $e^*$ in $G^*$ if and only if their corresponding faces $f$ and $g$ are separated by the edge $e$ in $G$. The graph $G^*$ is called the dual of $G$.

It is easy to see that the dual $G^*$ of a plane graph $G$ is planar; in fact, there is a natural way to embed $G^*$ in the plane. We place each vertex $f^*$ in the corresponding face $f$ of $G$, and then draw each edge $e^*$ in such a way that it crosses the corresponding edge $e$ of $G$ exactly once (and crosses no other edge of $G$). It is intuitively clear that we can always draw the dual as a plane graph in this way, but we shall not prove this fact. Note that if $e$ is a loop of $G$, then $e^*$ is a cut edge of $G^*$, and vice versa.




G G*

(a) (b)

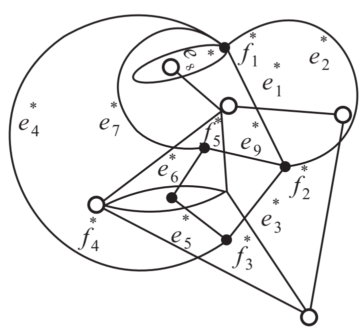



**Fig. 3.15: A Plane Graph and its Dual**

***Note:***

1. $G^*$ of a plane graph $G$ as a plane graph we can consider the dual $G^{**}$ of $G^*$ and $G^{**} \equiv G$.
2. Isomorphic plane graphs may have nonisomorphic duals.
3. The following relations are direct consequences of the definition of $G^*$
   (*i*) $v(G^*) = \phi(G)$
   (*ii*) $\in(G^*) = \in(G)$
   (*iii*) $d_G{}^*(f^*) = d_G(f)\ \forall f \in F(G)$.

## 3.7 Euler's Formula

Euler's formula states that if a finite, connected, planar graph is drawn in a plane without any edge intersection. If $v$ is the number of vertices, $e$ the number of edges and $f$, the number of faces (regions bounded by edges, including the outer, infinitely-large region), then

$$v - e + f = 2.$$

In a finite, connected, simple, planar graph, any face (except possibly the outer one) is bounded by at least three edges and every edge touches atmost two faces; using Euler's formula, we can show that these graphs are sparse in the sense that $e \leq 3v - 6$ if $v \geq 3$.

A simple graph is called maximal planar if it is planar but adding any edge would destroy that property. All the faces (even the outer one) are then bounded by three edges, explaining the alternative term triangular for these graphs. If a triangular graph has $v$ vertices with $v > 2$, then it has precisely $3v - 6$ edges and $2v - 4$ faces.

Euler's formula is also valid for simple polyhedra. This is no coincidence, every simple polyhedron can be turn onto a connected, simple, planar graph by using the polyhedron's edges as edges of the graph. The faces of the resulting planar graph corresponds to the faces of the polyhedron (Fig. 3.16).

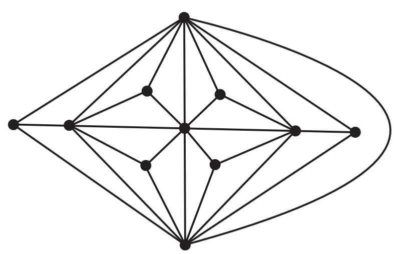


**Fig. 3.16: The Goldner–Harary Graph is Maximal Planer all its Faces are Bounded by three Edges**

### *Theorem 3.4*

*If G is a connected planar graph then $v - e + f = 2$.*

### *Proof*:

By induction we can prove on $f$, the number of faces in $G$. If $f = 1$, then each edge of $G$ is a cut edge and so $G$, being connected, is a tree. In this case $e = v - 1$. Suppose that it is true for all connected planar graphs with fewer than $n$ faces, and let $G$ be a connected planar graph with $n \geq 2$ faces. We choose an edge $e_1$ of $G$ that is not a cut edge.

Then $G - e_1$ is a connected planar graph and has $n - 1$ faces, since the two faces of $G$ separated by $e_1$ combine to form one face of $G - e_1$. From induction hypothesis, we have

$$v(G - e_1) - e(G - e_1) + f(G - e_1) = 2.$$

and, using the relations

$$v(G - e_1) = v(G)$$
$$e(G - e_1) = e(G) - 1$$
$$f(G - e_1) = f(G) - 1.$$

We set, finally

$$v(G) - e(G) + f(G) = 2.$$
$$\Rightarrow \quad v - e + f = 2.$$ ■

***Corollary 3.1:*** *All planar embeddings of a given connected planar graph have the same number of faces.*

### *Proof*:

Let $G$ and $H$ be two planar embeddings of a given connected planar graph.

Since $G \cong H$,

$v(G) = v(H)$ and $e(G) = e(H)$

using theorem 4, we have

$$f(G) = e(G) - v(G) = 2$$
$$= e(H) - v(H) + 2 = f(H)$$ ■

***Corollary 3.2:*** *If G is a simple planar graph with $v \geq 3$, then $e \leq 3v - 6$.*

### *Proof*:

It clearly suffices to prove this for connected graphs.

Let $G$ be a simple connected graph with $v \geq 3$.

Then $d(\phi_1) \geq 3$. for all $\phi_1 \in \phi$.

and $\sum_{\phi_1 \in \phi} d(\phi_1) \geq 3f$ (generating graph)

also $3e \geq 3f$.

From theorem 3.4,

$$v - e + \frac{2e}{3} \geq 2.$$

or $e \leq 3v - 6$ ■

***Corollary 3.3:*** *If G is a simple planar graph than $\delta \geq 5$.*

***Proof*:**

It is trivial for $v = 1, 2$

If $v \geq 3$, then from corollary 3.2, we have

$$\delta v \leq \sum_{v \in V} d(v) = 2e \sum 6v - 12.$$

It follows that $\delta \leq 5$. ■

***Corollary 3.4:*** *$K_5$ and $K_{3,3}$ are non-planar.*

***Proof*:**

If $K_5$ were planar than by corollary 3.2, we would have

$$10 = e(K_5) \leq 3v(K_5) - 6 = 9$$

This $K_5$ must be nonplanar

Suppose that $K_{3,3}$ is planar and let $G$ be a planar embedding of $K_{3,3}$

Since $K_{3,3}$ has no cycles of length less than four, every face of $G$ must have degree at least four.

$$4f \leq \sum_{\phi_1 \in \phi} d(\phi_1) = 2e = 18.$$

*i.e.* $f \leq 4$.

From theorem 3.4, we have

$$2 = v - e + f \leq 6 - 9 + 4 = 1.$$

Which is absurd. ■

## 3.8 Outerplanar Graphs

A graph is called outerplanar if it has an embedding in the plane such that the vertices lie on a fixed circle and the edges lie inside the disk of the circle and don't intersect. Equivalently, there is some face that includes every vertex. Every outerplanar graph is planar, but the converse is not true: the second example graph shown above ($K_4$) is planar but not outerplanar. This is the smallest non-outerplanar graph: a theorem similar to Kuratowski's states that a finite graph is outerplanar if and only if it does not contain a subgraph that is an expansion of

$K_4$ (the full graph on 4 vertices) or of $K_{2,3}$ (five vertices, 2 of which connected to each of the other three for a total of 6 edges).

### Properties of Outerplanar Graphs

- All finite or countably infinite trees are outerplanar and hence planar.
- An outerplanar graph without loops (edges with coinciding endvertices) has a vertex of degree at most 2.
- All loopless outerplanar graphs are 3-colorable; this face features prominently in the simplified proof of Chvatal's art gallery theorem by *Fisk* (1978). A 3-coloring may be found easily by removing a degree-2 vertex, coloring the remaining graph recursively, and adding back the removed vertex with a color different from its two neighbors.

### 3.8.1 *k-outerplanar Graphs*

*k*-outerplanar embedding of a graph is the same as an outerplanar embedding. For $k > 1$ a planar embedding is *k*-outerplanar if removing the vertices on the outer face results in a *k*-outerplanar embedding. A graph is *k*-outerplanar if it has a *k*-outerplanar embedding.

## 3.9 Solved Examples

■ **Example 3.2:** *A connected planar graph has 10 vertices each of degree 3. Into how many regions, does a representation of this planar graph split the plane?*

***Solution:***

Given that $n = 10$ and deg of each vertex = 3.

$\Rightarrow \quad \Sigma \deg(v) = 3 \times 10 = 30.$

But, $\quad \Sigma \deg(v) = 2e$

$\Rightarrow \quad 30 = 2e \Rightarrow e = 15$

Using Euler's formula, we have

$$n - e + f = 2$$

$\Rightarrow \quad 10 - 15 + f = 2$

$\Rightarrow \quad f = 7$

$\therefore \quad$ no of regions = 7. ■

■ **Example 3.3:** *Prove that $K_4$ and $K_{2,2}$ are planar*

***Solution:***

We have in $K_4$

$v = 4$ and $e = 6$

It is obvious that, $6 \leq 3*4 - 6 = 6$.

Hence relation is satisfied for $K_4$

For $K_{2,2}$, we have $v = 4$ and $e = 4$.

In this case, the related $e \leq 3v - 6$. that is, $4 \leq 3*4 - 6 = 6$ is satisfied.

Hence $K_4$ and $K_{2,2}$ are planar graphs. ■

■ **Example 3.4:** *If every region of a simple planar graph with n-vertices and e-edges embedded in a plane is bounded by K-edges then prove that*

$$e = \frac{K(n-2)}{K-2}$$

***Solution***:

We know that every region is bounded by $K$-edges, then $f$ regions are bounded by $Kf$-edges.

Each edge is counted twice, once for two of its adjacent regions.

We have $$2e = Kf$$

$$\Rightarrow \quad f = \frac{2e}{K} \quad \text{...(i)}$$

That is, if $G$ is a connected planar graph with $n$-vertices, $e$-edges and $f$-regions then

$$n - e + f = 2.$$

We have from ($i$)

$$n - e + \frac{2e}{K} = 2$$

$$\Rightarrow \quad nK - eK + 2e = 2K$$

$$\Rightarrow \quad nK - 2K = eK - 2e$$

$$\Rightarrow \quad K(n-2) = e(K-2)$$

$$\Rightarrow \quad e = \frac{K(n-2)}{K-2}$$ ■

■ **Example 3.5:** *If G is a graph with 1000 vertices and 3000 edges. What can you concluded about G is planar?*

***Solution***:

We know that, a graph is said to be planar if and only if it satisfies the inequality $q \leq 3p - 6$.

Here, we have,

$$p = 1000 \text{ and } q = 3000$$

Then, $3000 \leq 3p - 6.$

*i.e.* $3000 \leq 3000 - 6.$

$\Rightarrow$ $3000 \leq 2994$

which is not possible.

Hence we can conclude that $G$ is not a planar graph. ■

■ **Example 3.6:** *Find a graph G with degree sequence (4, 4, 3, 3, 3, 3) such that*

(*i*) *G is planar*

(*ii*) *G is not planar.*

***Solution***:

(*i*) We can draw a planar graph with six vertices (with degree sequence 4, 4, 3, 3, 3, 3.) as shown in Fig 3.17.

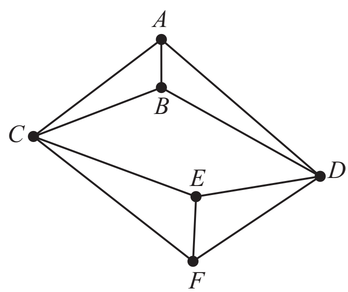



**Fig. 3.17**

(*ii*) Using Handshaking lamma, we have

$$\sum_{i=1}^{n} \deg(v_i) = 2q$$

$\Rightarrow$ $2q = 4 + 4 + 3 + 3 + 3 + 3$

$\Rightarrow$ $2q = 20$

$\Rightarrow$ $q = 10$

Hence the graph with $p = 6$, is planar iff it satisfy the inequality

$$q = 3p - 6$$

*i.e.* $10 \leq 3 \times 6 - 6$

$\Rightarrow$ $10 \leq 18 - 6$

$\Rightarrow$ $10 \leq 12.$ (which is not possible)

Hence it is not possible to draw a non-planar graph with given degree sequence 4, 4, 3, 3, 3, 3. ■

■ **Example 3.7:** *Show that every simple connected planar graph G with less than 12 vertices must have a vertex of degree ≤ 4.*

### *Solution*:

Let every vertex of $G$ has degree greater than or equal to 5

Then, if $d_1, d_2, d_3,\ldots,d_n$ are the degree of $n$ vertices of $G$.

We have $d_1 \geq 5, d_2 \geq 5, d_3 \geq 5,\ldots,d_n \geq 5$

So that $d_1 + d_2 + d_3 +,\ldots,+d_n \geq 5n$

Or $2m \geq 5n$ (Handshaking property)

$$\text{Or} \qquad \frac{5n}{2} \leq m \qquad \ldots(A)$$

On the other hand, we have, $m \leq 3n - 6$.

from ($A$), we have

$$\frac{5n}{2} \leq 3n - 6 \text{ or } n \geq 12$$

If every vertex of $G$ has degree ≥ 5,

then $G$ must have at least 12 vertices

Hence, if $G$ has less than 12 vertices, it must have a vertex of degree < 5. ■

■ **Example 3.8:** *What is the minimum number of vertices necessary for a simple connected graph with 11 edges to be planar?*

### *Solution*:

For a simple connected planar graph $G(m, n)$

we have, $m \leq 3n - 6$.

$$n \geq \frac{1}{3}(m+6)$$

when $m = 11$ we get $n \geq \frac{17}{3}$

Hence the required minimum number of vertices 6. ■

■ **Example 3.9:** *Let G be a planar connected graph with n-vertices, m edges and f regions and let its geometice dual G* have n* vertices, m* edges and f* regions, then prove that n* = f, m* = m and f* = n.*

*Further, it ρ and ρ* are the ranks and μ and μ* are the nullities of G and G* respectively, the prove that ρ* = μ and μ* = ρ.*

***Solution***:

Every region of $G$ yields exactly one vertex of $G^*$ and $G^*$ has no other vertex.

The number of regions in $G$ is precisely equal to the number of vertices of $G^*$

*i.e.* $$f = n^* \qquad ...(i)$$

Corresponding to every edge $e$ of $G$, there is exactly one edge $e^*$ of $G^*$ that crosses $e$ exactly once, and $G^*$ has no other edge.

Thus $G$ and $G^*$ have the same number of edges.

*i.e.* $$m = m^* \qquad ...(ii)$$

Using Euler's formula to $G^*$ and $G$, we get

$$\begin{aligned} f^* &= m^* - n - 2. \\ &= m - f + 2. \\ &= n. \end{aligned}$$

Since $G$ and $G^*$ are connected, we get

$$\rho = n - 1, \mu = m - n + 1$$
$$\rho^* = n^* - 1, \mu^* = m^* - n^* + 1.$$

These together with result ($i$) & ($ii$), and from Euler's formula

$$\begin{aligned} r^* = n^* - 1 &= f - 1 = (m - n + 2) - 1. \\ &= m - n + 1 = \mu. \\ \mu^* &= m^* - n^* + 1 = m - f + 1 \\ &= m - (m - n + 2) + 1 = n - 1. \\ &= \rho. \end{aligned}$$

■

■ **Example 3.10:** *If G is a simple connected planar graph s.f. G(p, q) having at least K edges in a boundary of each region. Then prove that (K – 2) q ≤ K (p – 2)*

***Solution***:

Every edge on the boundary of $G_1$ lies in the boundaries of exactly two regions of $G$.

$G$ may have some pendent edges which do not lie in a boundary of any region of $G$.

Sum of lengths of all boundaries of $G$ is less than twice the number of edges of $G$.

*i.e.* $$kf \leq 2q \qquad ...(i)$$

$G$ is a connected graph, by Euler's formula, we have

$$f = 2 + q - p \qquad ...(ii)$$

($ii$) – ($i$), we set

$$k(2 + q - p) \leq 2q$$

$$\Rightarrow \quad (K - 2)q \leq K(p - 2)$$ ■

■ **Example 3.11:** *Let G be a connected simple planar (n, m) graph is which every region is bounded by at least k edges. Show that* $m \leq \dfrac{k(n-2)}{(k-2)}$

***Solution***:

Every region in $G$ is bounded by at least $k$ edges.

We have $\quad 2m \geq kf \quad$ ....(i)

Where $f$ is the number of regions.

Substituting for $f$ from Euler's formula,

we get,

$$2m \geq k\,(m - n + 2)$$

$$\Rightarrow \quad K(n - 2) \geq km - 2m.$$

$$m \leq \frac{k(n-2)}{(k-2)}$$ ■

## SUMMARY

1. A path in a connected graph is **Eulerian** if it contains every edge exactly once.
2. A circuit in a connected graph is **Eulerian** if it contains every edge exactly once.
3. A connected graph **Eulerian** if it contains an Eulerian circuit.
4. A connected graph is Eulerian if and only if every vertex of the graph has even degree.
5. A connected graph contains an Eulerian path if and only if it has exactly two vertices of odd degree.
6. A graph is **planar** if it can be drawn in the plane, so not two edges cross.
7. $K_{3,3}$ and $K_5$ are nonplanar graphs.
8. **Euler's formula** If a connected planar graph with $e$ edges and $v$ vertices partitions the plane into $r$ regions, then $r = e - v + 2$.
9. **Kuratowski's theorem** A graph is planar if and only if it does not contain a subgraph homeomorphic to $K_5$ or $K_{3,3}$

## EXERCISES

1. Prove that a graph which contains a triangle can not be bipertite.
2. Show that $C_6$ is a bipertite graph.
3. Prove that the sum of the degrees of all regions in a map in equal to twice the number of edges in the corresponding path.
4. Draw all planar graphs with five vertices, which are not isomorphic to each other.
5. Show that of a planar graph $G$ of order $n$ and size $m$ has $f$ regions and $K$ components, then prove that $n - m = f = k + 1$
6. Let $G$ be a simple connected graph with fewer than 12 regions, in which each vertex has degree at least 3. Prove that $G$ has a region bounded by at most four edges.
7. Show that the condition $m \leq 3n - 6$ is not a sufficient condition for a connected simple graph with $n$ vertices and $m$ edges to be planar
8. What is the maximum number of edges possible in a simple connected planar graph with eight vertices? **[Ans.** 18**]**
9. Show that if $G$ is a plane triangulation, then $e = 3v - 6$.
10. Let $S = \{x_1, x_2, \ldots, x_n)$ be a set of $n \geq 3$ points in the plane such that the distance between any two points is at least one. Show that there are at most $3n - 6$ pairs of points at distance exactly one.

## Suggested Readings


1. **K. Apple** and **W. Haken**, "*Every Planar Map is 4-Colorable*" *Bulletin of the American Mathematical Society*, Vol. 82 (1976), pp. 711-712.
2. **J.A. Bondy** and **U.S.R. Murty**, *Graph Theory with Applications,* Elsevier, New York, 1976.
3. **R.A. Brualdi**, *Introductory Combinatorics*, 3rd ed., Prentice-Hall, Upper Saddle River, NJ, 1999.
4. **A. Tucker**, *Applied Combinatorics*, 2nd ed., Wiley, New York, 1984, pp. 3-79, 389-410.
5. **D.West**, *Introduction to Graph Theory,* 2nd ed., Prentice-Hall, Upper Saddle River, NJ, 2001.
6. **R.J. Wilson** and **J.J. Watkins**, *Graphs*: *An Introductory Approach,* Wiley, New York, 1990.


❑❑❑

# 4 Directed Graphs

**Edsger Wybe Dijkstra (1930-2002)**

Edsger Wybe Dijkstra (1930-2002) was born in Rotterdam. The Netherlands. He was a computer scientist and mathematician. He graduated from the university of Leyden in 1951 and subsiquently received his Ph.D. in 1956 from Leyden. In 1959 he was awarded by an honorary D.Sc. from Queen's University, Belfast.

Dijkstra taught mathematics at the Technical university, Nuenen, until 1973 and for next 11 years he had been a research fellow at Burroughs corporation, Nuenen. In 1984, he became the schlumberger centennial chair in computer science at the university of Texas, Austin.

Dijkstra was a distinguished fellow of British computer society. He made outstanding contributions to operating system, PL and Graph theory.

## 4.1 Introduction

There are so many problems lend themselves naturally to a graph-theoretic formulation. The concept of a graph is not quite adequate. While dealing with problems of traffic flow, it is necessary to know which roads in the network are one-way, and, in which direction the traffic is permitted. It is clear that, a graph of the network is not of much use in such a situation. We need in a graph, in which, each link has an assigned orientation - a *directed graph*.

*A directed graph D is an ordered triple (V(D), A(D),* $\psi_D$*) consisting of a nonempty set V(D) of vertices, a set A(D), disjoint from V(D), of arcs, and an incidence function* $\psi_D$ *which associates with each arc of D an ordered pair of vertices of D.*

If $a$ is an arc and $u$ and $v$ are vertices such that $\psi_D(a) = (u, v)$, then $a$ is said to join $u$ to $v$; $u$ is the tail of $a$, and $v$ is its head. We abbreviate "directed graph" to "digraph".

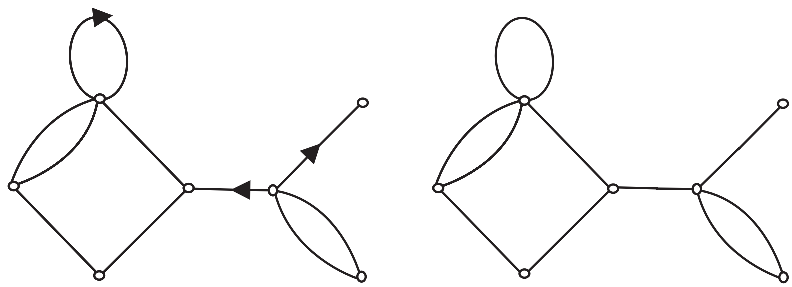


S. K. Yadav, *Advanced Graph Theory*,
https://doi.org/10.1007/978-3-031-22562-8_4

*A* digraph $D'$ is a subgraph of $D$ if $V(D') \subseteq V(D)$, $A(D') \subseteq A(D)$ and $\psi_D$ is the restriction of $\psi_D$ to $A(D')$.

Each digraph $D$ can be associated to a graph $G$ on the same vertex set; corresponding to each arc of $D$ there is an edge of $G$ with the same ends. This graph is called the underlying graph of $D$. As converse, given any graph $G$, we can obtain a digraph from $G$ by specifying, for each link, an order on its ends. This type of digraph is called an orientation of $G$.

Digraphs have a simple pictorial representation like graphs. A digraph is represented by a diagram of its underlying graph together with arrows on its edges, each arrow pointing towards the head of the corresponding arc (as shown in Fig. 4.1).



(a) (b)

**Fig. 4.1: (a) A Digraph *D*; (b) the Underlying graph *D***

All the concepts that are valid for graphs automatically applicable to digraphs too. However, there are many concepts that involve the notion of orientation, and these are applicable only to digraphs.

A directed 'walk' in $D$ is a finite non-null sequence $W = (v_0, a_1, v_1, ..., a_k, v_k)$, whose terms are alternately vertices and arcs, such that, for $i = 1, 2, ..., k$ the arc $a_1$ has head $v_1$ and tail $v_{i-1}$. As with walks in graphs, a directed walk $(v_0, a_1, v_1, ..., a_k, v_k)$ is represented simply by its vertex sequence $(v_0, v_1, ..., v_k)$.

A directed trail is a directed walk that is a trail; similarly, directed paths, directed cycles and directed tours are defined.

If there is a directed $(u, v)$-path in $D$, vertex $v$ is said to be reachable from vertex $u$ in $D$. Two vertices are diconnected in $D$ if each is reachable from the other. As in the case of connection in graphs, diconnection is an equivalence relation on the vertex set of $D$. The subdigraphs $D[V_1]$, $D[V_2], ..., D[V_m]$ induced by the resulting partition $(V_1, V_2, ..., V_m)$ of $V(D)$ are called the dicomponents of $D$. A digraph $D$ is diconnected if it has exactly one dicomponent. (see Fig. 4.2)

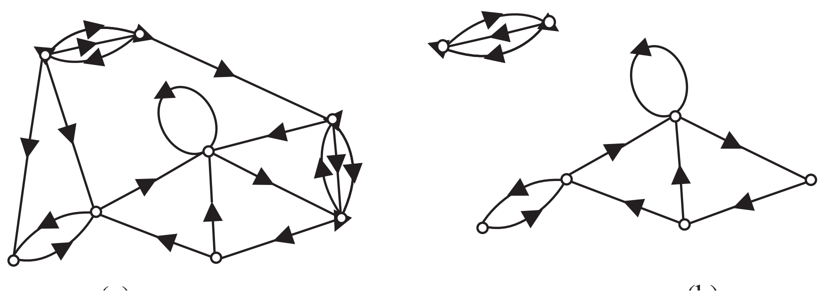


**Fig. 4.2: The Digraph of Fig. (a) is not Disconnected and Fig. (b) is has three Dicomponents**

The indegree $d_D^-(v)$ of a vertex $v$ in $D$ is the number of arcs with head $v$; the $d_D^+(v)$ outdegree of $v$ is the number of arcs with tail $v$. We denote the minimum and maximum indegrees and outdegrees in $D$ by $\delta^-(D)$, $\Delta^-(D)$, $\delta^+(D)$ and $\Delta^+(D)$, respectively. A digraph is strict if it has no loops and no two arcs with the same ends have the same orientation. Throughout this chapter, $D$ will denote a digraph and $G$ its underlying graph. This is a useful convention; it allows us, for example, to denote the vertex set of $D$ by $V$ (since $V = V(G)$), and the numbers of vertices and arcs in $D$ by $v$ and $\varepsilon$, respectively. Also, as with graphs, we shall drop the letter $D$ from our notation whenever possible; thus we write $A$ for $A(D)$, $d^+(v)$ for$d_D^+(v)$ $\delta^-$ for $\delta^-(D)$, and so on.

## 4.2 Directed Paths

In a digraph, There is no close relationship between the lengths of paths and directed paths. We can see from Fig. 4.3, which has no directed path of length greater than one.

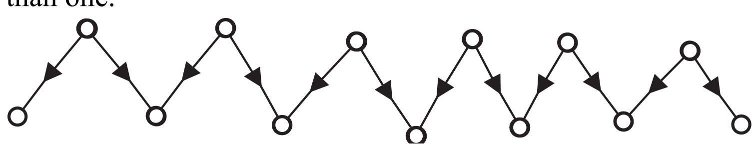


**Fig. 4.3**

Some information about the length of denoted paths in a digraph can be obtained by looking chromatic numbers.

### *Theorem 4.1: (Ray and Gallai, 1967-68)*

*A digraph D contains a directed path of length X–1*.

### *Proof:*

We consider $A'$ be a minimal set of arcs of $D$ such that $D' = D - A'$ containing no directed cycle and the length of a longest directed path in $D'$ be $k$. We assign

colours 1, 2, 3..., $k + 1$ to the vertices of $D'$ by assigning colour $i$ to vertex $v$ if the length of a longest directed path in $D'$ with origin $v$ is $(i - 1)$.

Denoting by $V_i$ the set of vertices with colour $i$ we can show that $(V_1, V_2, \ldots, V_{k+1})$ is a proper $(k + 1)$ vertex colouring of $D$.

We can observe that the origin and terminus of any directed path in $D'$ have different colours. For Let $P$ be a directed $(u, v)$ – path of positive length in $D'$ and suppose $v \in V_i$. Then, there exists a directed path $Q = (v_1, v_2, v_3 \ldots, v_i)$ in $D'$, where $v_i = v$.

Since $D'$ contains no directed cycle, $PQ$ is a directed path with origin $u$ and length at least $i$,

Thus $u \notin V_i$.

We are to show now, the ends of any arc of $D$ have different colours.

Suppose $(u, v) \in A(D)$.

If $(u, v) \in A(D')$ then $(u, v)$ is a directed path in $D'$ and so $u$ and $v$ have different colours.

Otherwise, $(u, v) \in A'$.

By the minimality of $A'$, $D' + (u, v)$ contains a directed cycle $C$.

$C - (u, v)$ is a directed $(u, v)$ – path in $D'$ and hence in this case, $u$ and $v$ have different colours.

The above concludes, that $(V_1, V_2, V_3 \ldots, V_{k+1})$ is a proper vertex colouring of $D$.

This follows that

$X \leq k + 1$, and so $D$ has a directed path of length $k \geq X - 1$. ■

***Note:***

*(i) The theorem 4.1 is a best possible way to show that every graph G has an orientation in which the longest directed path is of length $X - 1$.*

*(ii) Given a proper $X$ – vertex colouring $(V_1, V_2, \ldots, V_x)$ of G, we orient G by converting edge uv to are $(u, v)$ if $u \in V_i$ and $v \in V_j$ with $i < j$.*

## 4.3 Tournament

An orientation of a complete graph is termed as tournament. The tournaments of four vertices are shown in Fig. 4.4. Each of them is to be regarded as indicating the results of games in a round-robin tournament between four players; *e.g.* the first tournament in Fig. 4.4 is shown that one player has won all three games and that the other three have each won one. A directed Hamiltonian path of $D$ is a directed path that includes every vertex of $D$.

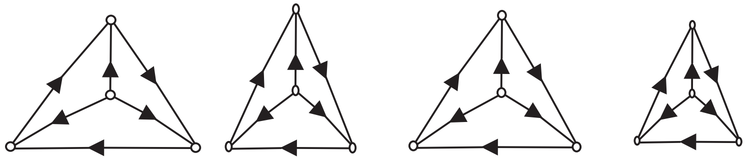


**Fig. 4.4 The tournament of four vertices**

### *Theorem 4.2*

*Every tournament has a directed Hamiltonian path.*

### *Proof:*

Every tournament has a path such that $D$ is directed. It includes every vertex of $D$ then If $D$ is a tournament, then

$X = v$. ■

### *Note:*

(i) *An interesting fact about tournament is that there is always a vertex from which every other vertex can be reached in at most two steps. An in-neighbour of vertex v in D is a vertex u such that $(u, v) \in A$; and an out neighbour of v is a vertex w such that $(v, w) \in A$.*

(ii) *We denote the sets of in-neighbours and out-neighbours of v in D $d_D^-(v)$ $d_D^+(v)$ by and respectively.*

## *Theorem 4.3*

*A loopless digraph D has an independent set S such that each vertex of D not in S is reachable from a vertex in S by a directed path of length at most two.*

### *Proof:*

This theorem can be proved by induction on $v$. The theorem holds trivially for $v = 1$.

We assume that it is true for all digraphs with fewer than $v$ vertices, and let $v$ be an arbitrary vertex of $D$.

By induction hypothesis there exists in $D' = D - (\{v\} \cup N^+(v))$ an independent set $S'$ such that each vertex of $D'$ not in $S'$ is reachable from a vertex is $S'$ by a directed path of length at most two. If $v$ is an out-neighbour of some vertex $u$ of $S'$, then every vertex of $N^+(v)$ is reachable from $u$ by a directed path of length two.

Hence, in this case, $S = S'$ satisfies the required property.

One the other hand, $v$ is not an out-neighbour of any vertex of $S'$, then $v$ is joined to no vertex of $S'$ and the independent set. $S = S' \cup \{v\}$ has the result.

■

***Note:***

*A tournament contains a vertex from which every other vertex is reachable by a directed path of length at most two, for this, if D is a tournament, then* $\alpha = 1$*:*

## 4.4 Directed Cycles

Theorem 4.2 states that every tournament contains a directed Hamiltonium path. It includes that, if the tournament is assumed to be disconnected.

If $S$ and $T$ are subsets of $V$, we denote by $(S, T)$ the set of arcs of $D$ that have their tails in $S$ and their heads in $T$.

### *Theorem 4.4: (Moon's and Dirac's Theorem) (1966)*

*Each vertex of a diconnected tournament D with* $v \geq 3$ *is contained in a directed* $k$ *– cycle,* $3 \leq k \leq v$

### *Proof:*

Let $D$ be a disconnected tournament with $v \geq 3$ let $u$ be any vertex of $D$.

Set $S = N^{+}(u)$ and $T = N^{-}(u)$.

We show that $u$ is in a directed 3 – cycle.

Since $D$ is diconnected, neither $S$ nor $T$ can be empty; and, for the same reason, $(S, T)$ must be non-empty (As shown in fig 4.5). There exists some arc $(v, w)$ in $D$ with $v \in S$ and $w \in T$, and $u$ is in the directed 3 – cycle $(u, v, w, u)$.

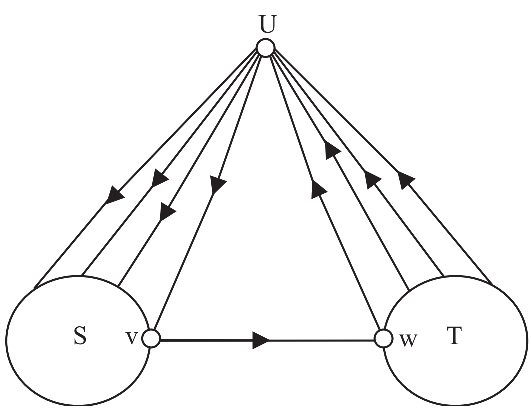



**Fig. 4.5: Diconnected Tournament**

The theorem can now be proved by induction on $k$.

Suppose, $u$ is in directed cycle of all lengths between 3 and $n$, where $n < v$. We can show that $u$ is in directed $(n + 1)$ – cycle.

We consider $C = (v_0, v_1, v_2,...,v_n)$ be a directed $n$ – cycle in which $v_0 = v_n = u$.

If there is a vertex in $V(D)\backslash V(C)$ which is both the head of an arc with tail in $C$ and the tail of an arc with head in $C$, then there are adjacent vertices $v_i$ and $v_{i+1}$ on $C$ s.t. both $(v_i, v)$ and $(v, v_{i+1})$ are arcs of $D$. In this case $u$ is in the directed $(n + 1)$ – cycles. $(v_0, v_1, v_2,...,v_i, v_1, v_{i+1},...,v_n)$.

Otherwise, denoting by $S$ the set of vertices in $V(D)\backslash V(C)$ which are heads of arcs joined to $C$, and by $T$ the set of vertices in $V(D)\backslash V(C)$ which are tails of arcs joined to $C$. (as shown in Fig. 4.6).

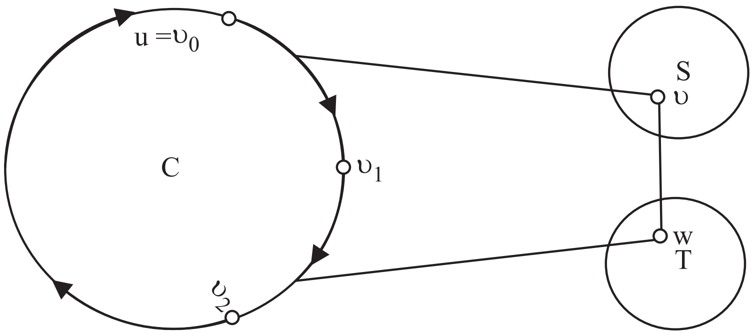



**Fig. 4.6**

Since $D$ is diconnected, $S$, $T$ and $(S, T)$ are all nonempty, and there is some arc $(v, w)$ in $D$ with $v \in S$ and $w \in T$.

Hence $u$ is in the directed $(n + 1)$ – cycle $(v_0, v, w, v_2,...,v_n)$. ■

***Note:***

*A directed Hamiltonian cycle of D is a directed cycle that includes every vertex of D.*

## *Theorem 4.5 (Ghouila – Houri's Theorem) (1960)*

*If D is strict and min* $\{\delta^-, \delta^+\} \geq \frac{v}{2} > 1$*, then D contains a directed Hamiltonian cycle.*

### *Proof:*

Let $D$ satisfies the hypotheses of the theorem, but does not contain a directed Hamiltonian cycle.

We denote the length of the longest directed cycle in $D$ by $l$, and let $C = (v_1, v_2,..., v_l, v_1)$ be a directed cycle in $D$ of length $l$.

We note that $l > \frac{v}{2}$.

Let $P$ be a longest directed path in $D - V(C)$ and suppose that $P$ has origin $u$, terminus $v$ and length in (as shown in Fig. 4.7).

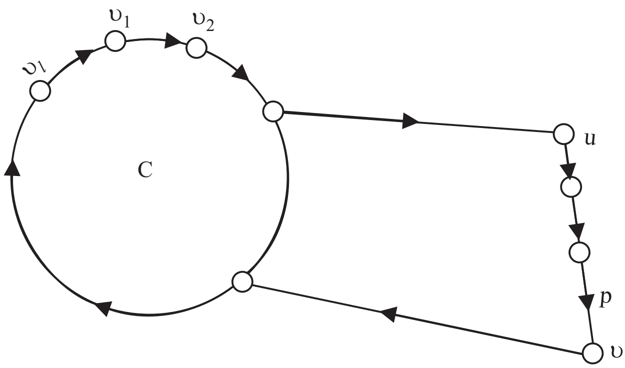



**Fig. 4.7**

It is clear that,

$$v \geq l + m + 1 \quad \text{... }(i)$$

since $l > \dfrac{v}{2}$

$$\Rightarrow \qquad m < \frac{v}{2} \quad \text{... }(ii)$$

Set $S = \{i \,|(v_{i-1}, u) \in A\}$ and $T = \{i \,|(v, v_i) \in A\}$

We show that $S$ and $T$ are disjoint.

Let $C_{j,\,k}$ denote the section of $C$ with origin $v_j$ and terminus $v_k$.

If some integer $i$ were in both $S$ and $T$, $D$ would contain the directed cycle $C_{i,\,i-1}$ $(v_{i-1}, u)$ $P(v, v_i)$ of length $l + m + 1$, contradicting the choice of $C$

Then, $S \cap T = \phi$ ... $(iii)$

Now, because $P$ is a maximal directed path in

$D - V(C)$, $N^-(u) \subseteq V(P) \cup V(C)$.

But the number of in - neighbours of $u$ in $C$ is precisely $|S|$ and so

$d_D^-(u) = d_P^-(u) + |S|$

since $d_D^-(u) \geq \delta \geq \dfrac{v}{2}$ and $d_P^-(u) \leq m$,

$$\Rightarrow \qquad |S| \geq \frac{v}{2} - m \quad \text{....}(iv)$$

Similarly,

$$|T| \geq \frac{v}{2} - m \quad \text{....}(v)$$

by $(ii)$, both $S$ and $T$ are nonempty.

Adding equ. ($iv$) and ($v$) and using ($i$), we get

$$|S| + |T| \geq l - m + 1.$$

Therefore, by ($iii$)

$$|S \cup T| \geq l - m + 1 \qquad ....(vi)$$

since $S$ and $T$ are disjoint and nonempty, there are positive integers $i$ and $k$ s.t. $i \in S$, $i + k \in T$.

and

$$i + j \notin S \cup T \text{ for } i \subseteq j \subset k \qquad .....(vii)$$

from ($vi$) & ($vii$), it can be seen that, $k \leq$ m.

Thus the directed cycle

$C_{i+k,\ i-1}(v_{i+1}, u)\, P(v, v_{i+k})$, which has length $l + m + 1 - k$, is longer than $C$.

Which is a contradiction to establish the theorem. ■

## 4.5 Acyclic Graph

*An orientation D of an undirected graph G is said to be acyclic, if and only if it has no directed cycle.*

$\alpha(G)$ be denoted the number of acyclic orientation of $G$.

### *Theorem 4.6: (Stanley Theorem, 1973)*

*Let G be a graph of order n. Then the number of the acylic orientations of G is*

$$\alpha(G) = (-1)^n \chi_G(-1)$$

Where $\chi_G$ is the chromatic polynomial of $G$.

### *Proof:*

This proof is possible by induction on $\in_G$.

Firstly, if $G$ is discrete, then $\chi_G(k) = k^n$, and $\alpha(G) = 1 = (-1)^n (-1)^n = (-1)^n \chi_G(-1)$ as required Now $\chi_G(k)$ is a polynomial that satisfies the recurrence $\chi_G(k) = \chi_G - e(k) - \chi_{G*e}(k)$. To prove the claim, we have to show that $\alpha(G)$ satisfy the same recurrence.

If $\qquad \alpha(G) = \alpha(G - e) + \alpha(G * e) \qquad ....(i)$

Then, by induction hypothesis,

$$\begin{aligned} \alpha(G) &= (-1)^n \chi_{G-e}(-1) + (-1)^{n-1} \chi_{G*e}(-1) \\ &= (-1)^n \chi_G(-1) \end{aligned}$$

for equ. ($i$) we can see that every acyclic orientation of $G$ gives an acylic orientation of $G - e$. On the other hand, if $D$ is an acyclic orientation of $G - e$ for $e = u\,v$, it extends to an acylic orientation of $G$ by putting $e_1 : u \rightarrow v$ or $e_2 : v \rightarrow u$.

If $D$ has no directed path $u \xrightarrow{*} v$, we choose $e_2$ and if $D$ has no acyclic, it cannot have both ways $u \xrightarrow{*} \text{v}$ and $v \xrightarrow{*} u$

We can conclude that

$$\alpha(G) = \alpha(G - e) + b,$$

where $b$ is the number of acyclic orientation $D$ of $G - e$ that extend in both ways $e_1$ and $e_2$. The acyclic orientation $D$ that extend in both ways are exactly those that contain.

Neither $u \xrightarrow{*} v$ nor $v \xrightarrow{*} u$ as a directed path ...(*ii*)

Each acyclic orientation of $G * e$ corresponds in a natural way to an acyclic orientation $D$ of $G - e$ that satisfies (*ii*).

Therefore $\qquad b = a\,(G * e)$

■

## 4.6 Di-Orientable Graph

A Graph $G$ is said to be di-orientable, if and only if there is a di-connected oriented graph $D$ such that $G = U(D)$

### *Theorem 4.7: (Robbins Theorem, 1939)*

*A connected graph G is diorientable if and only if G has no bridges.*

### *Proof:*

If G has a bridge $e$, then any orientation of $G$ has atleast two dicomponents.

Suppose that $G$ has no bridges, Hence $G$ has a cycle $C$, and $a$ cycle is always di-orientable.

Let $H \subseteq G$ be maximal such that it has a di-oriented $D_H$. If $H = G$, then it has done.

Otherwise, there exists an edge $e = v^u \in E_G$ such that $u \in H$ but $v \notin H$ (because $G$ is connected).

The edge $e$ is not a bridge and thus there exists a cycle.

$$C' = \text{e}\,PQ : v \to u \xrightarrow{*} w \xrightarrow{*} v$$

in $G$, where $w$ is the last vertex inside $H$

We can observe in Fig. 4.8. In the di-orientation $D_H$ of $H$ there is a directed path

$$P' : u \xrightarrow{*} w.$$

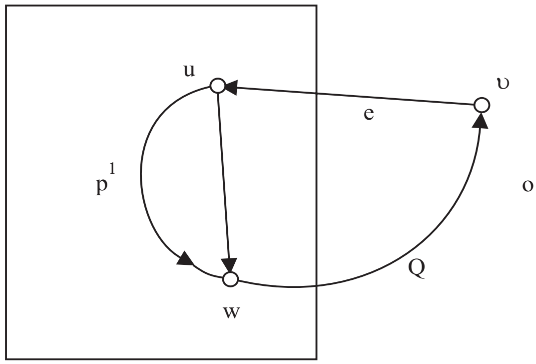



**Fig. 4.8**

We can orient $e : v \to u$ and the edges of $Q$ in the direction

$Q : w \xrightarrow{*} v$ to obtain a directed

cycle $\quad e\,P'Q : v \to u \xrightarrow{*} w \xrightarrow{*} v.$

We can conclude that, $G[V_H \cup V_C]$ has a di-orientation, which contradicts the maximality assumption on $H$. ■

### *Note*:

*Let D be a directed graph. A directed Euler tour of D is a directed closed walk that uses each edge exactly once. A directed Euler trail of D is a directed walk that uses each edge exactly once.*

***Example 4.1:*** Consider a tournament of six teams 1, 2, 3,…,6. and let $T$ be the scoring digraph as in the Fig. 4.9. Here $1 \to 2 \to 4 \to 5 \to 6 \to 3$ is a directed Hamilton path, but this extends to a directed Hamilton Cycle (by adding $3 \to 1$) so, for every team there is a Hamilton path, where it is a winner, and in another, it is a looser?

### *Solution:*

We consider, $s_1(j) = d_T(j)$ be the winning number of the team $j$ (the number of teams beaten by $j$).

In the above tournament

$s_1(1) = 4$, $s_1$ $(2) = 3$, $s_1(3) = 3$, $s_1(4) = 2$, $s_1(5) = 2$, $s_1(6) = 1$.

so, is the team 1 a winner? If so, is 2 or 3 next?

We define the second level scoring for each team by

$$s_2(j) = \sum_{ji \in E_T} s_1(i) \qquad \text{... } (i)$$

This tells us how good team $j$ beat.

In the above figure, we have

$s_2(1) = 8$, $s_2(2) = 5$, $s_2(3) = 9$, $s_2(4) = 3$, $s_2(5) = 4$ and $s_2(6) = 3$.

Now, it seems that 3 in the winner, but 4 and 6 have the same score. We can continue by defining inductively the m$^{\text{th-}}$level scoring by

$$s_m(j) = \sum_{ji \in E_T} s_{m-1}(i) \qquad \text{... } (ii)$$

It can be proved by matrix method that for a di-connected tournament with atleast four teams, the level scorings will eventually stabilize in a ranking of the tournament; there exists an m for which the mth level scoring gives the same ordering as do the $(m + k)$th-level scorings for all $k \geq 1$.

If $T$ is not di-connected;, then the level scoring should be carried out with respect to the di-components.

In the example the level scoring gives

$$1 \rightarrow 3 \rightarrow 2 \rightarrow 5 \rightarrow 4 \rightarrow 6$$

as the ranking of the tournament.

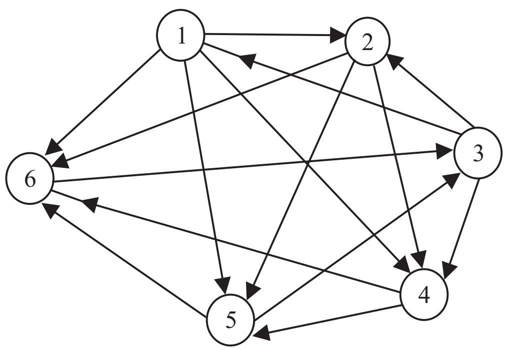



**Fig. 4.9**

## *Theorem 4.8*

*If G is a 2-edge-connected, then G has a di-connected orientation.*

### *Proof:*

Let $G$ be 2-edge-connected. Then $G$ contains a cycle $G_1$. We define inductively a sequence $G_1, G_2, \ldots,$ of connected subgraphs of $G$ as follows: if $G_i$ $(i = 1, 2, \ldots)$ is not a spanning subgraph of $G$, let $v_i$ be a vertex of $G$ not in $G_i$. Then there exist edge-disjoint paths $P_i$ and $Q_i$ from $v_i$ to $G_i$. Define

$$G_{i+1} = G_i \cup P_i \cup Q_i$$

Since $v(G_{i+1}) > v(G_i)$, this sequence must terminate in a spanning subgraph $G_n$ of $G$.

We now orient $G_n$ by orienting $G_1$ as a directed cycle, each path $P_i$ as a directed path with origin $v_i$, and each path $Q_i$ as a directed path with terminus $v_i$. Clearly every $G_i$, and hence in particular $G_n$, is thereby given a diconnected orientation. Since $G_n$ is a spanning subgraph of $G$ it follows that $G$, too, has a diconnected orientation.

### Theorem 4.9

*Let G be a 2k-edge-connected graph with an Euler trail. Then G has a k-arc-connected orientation.*

### Proof:

Let $v_0e_1v_1,\ldots,e_\varepsilon v_\varepsilon$ be an Euler trail of $G$. Orient $G$ by converting the edge $e_i$ with ends $v_{i-1}$ and $v_i$ to an arc $a_i$ with tail $v_{i-1}$ and head $v_i$, for $1 \leq i \leq \varepsilon$. Now let $[S, \bar{S}]$ be an $m$-edge cut of $G$. The number of times the directed trail $(v_0, a_1, v_1, \ldots, a_\varepsilon, v_\varepsilon)$ crosses from $S$ to $\bar{S}$ differs from the number of times it crosses from $\bar{S}$ to $S$ by at most one. Since it includes all arcs of $D$, both $(S, \bar{S})$ and $(\bar{S}, S)$ must contain at least $[m/2]$ arcs.

■

### Theorem 4.10

*Let D be a di-connected tournament with $v \geq 5$, and let A be the adjacency matrix of D. Then $A^{d+3} > 0$ where d is the directed diameter of D.*

### Proof:

The $(i, j)$th entry of $A^k$ is precisely the number of directed $(v_i, v_j)$-walks of length $k$ in $D$. We must therefore show that, for any two vertices $v_i$ and $v_j$ (possibly identical), there is a directed $(v_i, v_j)$-walk of length $d + 3$.

Let $$d_{ij} = d(v_i, v_j)$$

Then $0 \leq d_{ij} \leq d \leq v - 1$ and therefore

$$3 \leq d - d_{ij} + 3 \leq v + 2$$

If $d - d_{ij} + 3 \leq v$ then, there is a directed $(d - d_{ij} + 3)$-cycle $C$ containing $v_j$. A directed $(v_i, v_j)$-path $P$ of length $d_{ij}$ followed by the directed cycle $C$ together form a directed $(v_i, v_j)$-walk of length $d + 3$, as desired.

There are two special cases. If $d - d_{ij} + 3 = v + 1$, then $P$ followed by a directed $(v - 2)$-cycle through $v_i$ followed by a directed 3-cycle through $v_i$ constitute a directed $(v_i, v_j)$-walk of length $d + 3$ (the $(v - 2)$-cycle exists since $v \geq 5$); and if $d - d_{ij} + 3 = v + 2$, then $P$ followed by a directed $(v - 1)$-cycle through $v_i$ followed by a directed 3-cycle through $v_i$ constitute such a walk.

■

## 4.7 Applications of Directed Graphs

### 4.7.1 *Job Sequencing Problem*

Let a number of jobs $J_1, J_2, J_3, \ldots, J_n$ have to be processed on a machine; for example, each $J_i$ might be an order of bottles or jars in a glass factory. After each job the machine must be adjusted to fit the requirements of the next job.

If the time of adaptation from job $J_i$ to job $J_i$ is $t_{ij}$, find a sequencing of the jobs that minimises the total machine adjustment time.

This problem is clearly related to the travelling salesman problem, and no efficient method for its solution is known. It is therefore desirable to have a method for obtaining a reasonably good (but not necessarily optimal) solution.

***Step 1:*** Construct a digraph $D$ with vertices $v_1, v_2, \ldots, v_n$, such that $(v_i, v_j) \in A$ if and only if $t_{ij} \leq t_{ji}$. By definition, $D$ contains a spanning tournament.

***Step 2:*** Find a directed Hamilton path $(v_{i1,}\ v_{i2}, \ldots, v_{in})$ of $D$, and sequence the jobs accordingly.

Since step 1 discards the larger half of the adjustment matrix $[t_{ij}]$, it is a reasonable supposition that this method, in general, produces a fairly good job sequence. Note, however, that when the adjustment matrix is symmetric, the method is of no help whatsoever.

As an example, suppose that there are six jobs $J_1, J_2, J_3, J_4, J_5$ and $J_6$ and that the adjustment matrix is

| | $J_1$ | $J_2$ | $J_3$ | $J_4$ | $J_5$ | $J_6$ |
|---|---|---|---|---|---|---|
| $J_1$ | 0 | 5 | 3 | 4 | 2 | 1 |
| $J_2$ | 1 | 0 | 1 | 2 | 3 | 2 |
| $J_3$ | 2 | 5 | 0 | 1 | 2 | 3 |
| $J_4$ | 1 | 4 | 4 | 0 | 1 | 2 |
| $J_5$ | 1 | 3 | 4 | 5 | 0 | 5 |
| $J_6$ | 4 | 4 | 2 | 3 | 1 | 0 |

The sequence $J_1 \to J_2 \to J_3 \to J_4 \to J_5 \to J_6$ requires 13 units in adjustment time. To find a better sequence, construct the digraph $D$ as in step 1. (Fig. 4.10)

$(v_1, v_6, v_3, v_4, v_5, v_2)$ is a directed Hamilton path of D, and yields the sequence

$$J_1 \to J_6 \to J_3 \to J_4 \to J_5 \to J_2.$$

which requires only eight units of adjustment time. Note that the reverse sequence

$$J_2 \to J_5 \to J_4 \to J_3 \to J_6 \to J_1$$

is far worse, requiring 19 units of adjustment time.

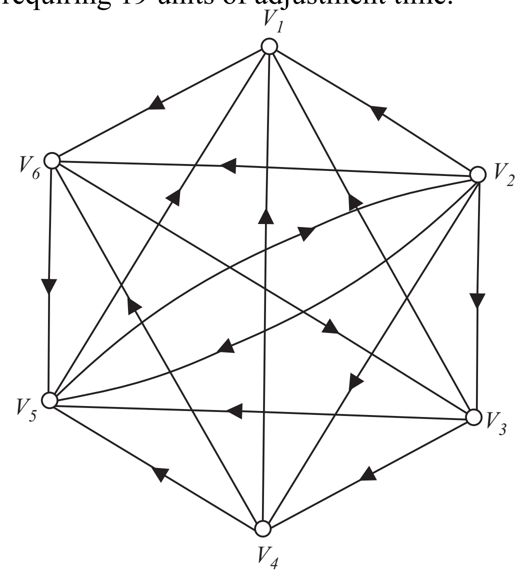



**Fig. 4.10: Segnential Digraph**

## 4.7.2 *To Design an Efficient Computer Drum*

The position of a rotating drum is to be recognised by means of binary signals produced at a number of electrical contacts at the surface of the drum. The surface is divided into $2^n$ sections, each consisting of either insulating or conducting material. An insulated section gives signal 0 (no current), whereas a conducting section gives signal 1 (current). For example, the position of the drum in Fig .4.11 gives a reading 0010 at the four -contacts.

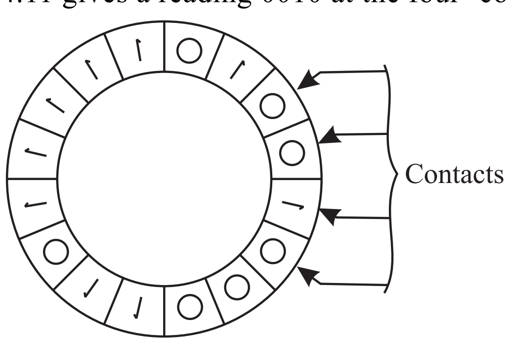
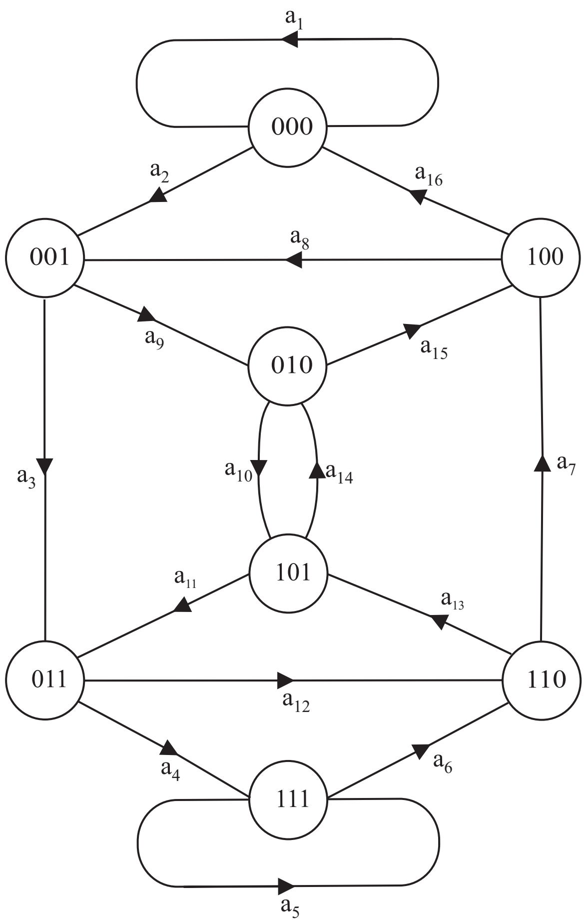



**Fig. 4.11: A Computer Drum**




| Arc | Label |
|---|---|
| $a_1$ | 0000 |
| $a_2$ | 0001 |
| $a_3$ | 0011 |
| $a_4$ | 0111 |
| $a_5$ | 1111 |
| $a_6$ | 1110 |
| $a_7$ | 1100 |
| $a_8$ | 1001 |
| $a_9$ | 0010 |
| $a_{10}$ | 0101 |
| $a_{11}$ | 1011 |
| $a_{12}$ | 0110 |
| $a_{13}$ | 1101 |
| $a_{14}$ | 1010 |
| $a_{15}$ | 0100 |
| $a_{16}$ | 1000 |

**Fig. 4.12: Binary Form**

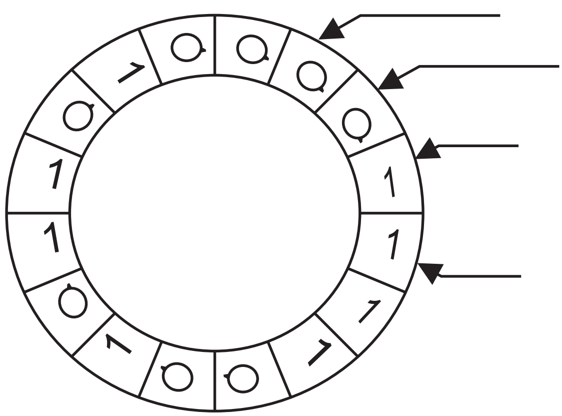



**Fig. 4.13: Signential Drum**

If the drum were rotated clockwise on section, the reading would be 1001. Thus these two positions can be distinguished, since they give different readings. However, a further rotation of two sections would result in another position with reading 0010, and therefore this latter position is indistinguishable from the initial one.

We design the drum surface in such a way that the $2^n$ different positions of the drum can be distinguished by $k$ contacts placed consecutively around part of the drum, and we would like this number $k$ to be as small as possible.

First note that $k$ contacts yield a $k$-digit binary number, and there are $2^k$ such numbers. Therefore, if all $2^n$ positions are to give different readings, we must have $2^k \geq 2^n$, that is, $k \geq n$. We shall show that the surface of the drum can be designed in such a way that $n$ contacts suffice to distinguish all $2^n$ positions.

We define a digraph $D_n$ as follows: the vertices of $D_n$ are the $(n - 1)$-digit binary numbers $p_1p_2,\ldots, p_{n-1}$ with $p_i = 0$ or 1. There is an arc with tail $p_1p_2,\ldots, p_{n-1}$ and head $q_1q_2,\ldots,q_{n-1}$ if and only if $p_{i+1} = q_i$ for $1 \leq i \leq n-2$; in other words, all arcs are of the form $(p_1p_2,\ldots, p_{n-1}, p_2p_3,\ldots, p_n)$. In addition, each arc $(p_1p_2\ldots p_{n-1}, p_2p_3,\ldots, p_n)$ of $D_n$ is assigned the label $p_1p_2,\ldots, p_n$. $D_4$ is shown in Fig. 4.12

Clearly, $D_n$ is connected and each vertex of $D_n$ has indegree two and outdegree two. Therefore, $D_n$ has a directed Euler tour. This directed Euler tour, regarded as a sequence of arcs of $D_n$, yields a binary sequence of length $2^n$ suitable for the design of the drum surface.

For example, the digraph $D_4$ of Fig. 4.12 has a directed Euler tour $(a_1, a_2,\ldots, a_{16})$, giving the 16-digit binary sequence 0000111100101101. (Just read off the first digits of the labels of the $a_i$.) A drum constructed from this sequence is shown in Fig. 4.13.

### 4.7.3 Ranking of the Participants in a Tournament

A number of players each play one another in a tennis tournament. Given the outcomes of the games, how should the participants be ranked?

We consider, for example, the tournament of Fig. 4.13. This represents the result of a tournament between six players; we see that player 1 beat players 2, 4, 5 and 6 and lost to player 3, and so on.

One possible approach to ranking the participants would be to find a directed Hamilton path in the tournament, and then rank according to the position on the path. For instance, the directed Hamilton path (3, 1, 2, 4, 5, 6) would declare

player 3 the winner, player 1 runner-up, and so on. This method of ranking, does not bear further examination, since a tournament generally has many directed Hamilton paths; our example has (1, 2, 4, 5, 6, 3), (1, 4, 6, 3, 2, 5) and several others.

Another approach would be to compute the scores (numbers of games won by each player) and compare them. We obtain the score vector as:

$$s_1 = (4, 3, 3, 2, 2, 1)$$

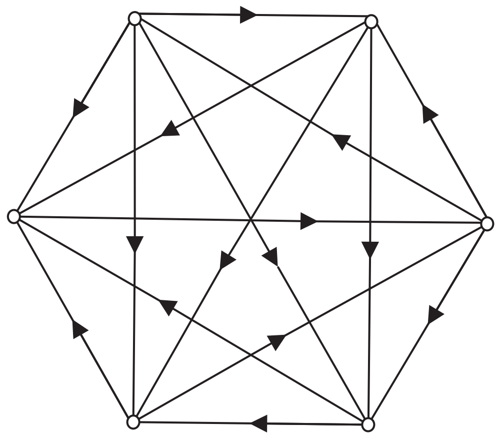


**Fig. 4.13: Ranking Problem**

This score vector does not distinguish between players 2 and 3 even though player 3 beat players with higher scores than did player 2. We are thus led to the second-level score vector

$$s_2 = (8, 5, 9, 3, 4, 3)$$

in which each player's second-level score is the sum of the scores of the players he beat. Player 3 now ranks first. Continuing this procedure we obtain further vectors as under:

$$s_3 = (15, 10, 16, 7, 12, 9)$$

$$s_4 = (38, 28, 32, 21, 25, 16)$$

$$s_4 = (90, 62, 87, 41, 48, 32)$$

$$s_6 = (183, 121, 193, 80, 119, 87)$$

The ranking of the players is seen to fluctuate a little, player 3 vying with player 1 for first place. This procedure always converges to a fixed ranking when the tournament in question is disconnected and has at least four vertices. This will lead to a method of ranking the players in any tournament.

***Note*:**

*In a diconnected digraph D, the length of a shortest directed (u, v)-path is denoted by $d_D(u, v)$ and is called the distance from u to v; the directed diameter of D is the maximum distance from any one vertex of D to any other.*

## 4.8 Network Flows

Various transportation networks or pipelines are conveniently represented by weighted directed graphs. These networks possess some additional requirements. Goods are transported through specific places or warehouses to final locations or market places through a network of roads.

A network N consists of:

(*i*) An underlying digraph $D = (V, E)$

(*ii*) Two distinct vertices $s$ and $r$, known as the source and the sink of $N$.

(*iii*) A capacity function $A : V \times V \to IR_+$ for which

$$\alpha(e) = 0, \text{ iff } e \notin E.$$

We denote $V_N = V$ and $E_N = E$.

Let $A \subseteq V_N$ be a set of vertices

and $f : V_N \times V_N \to IR$ any function such that $f(e) = 0$ if $e \notin E_N$.

We can adopt the following notations:

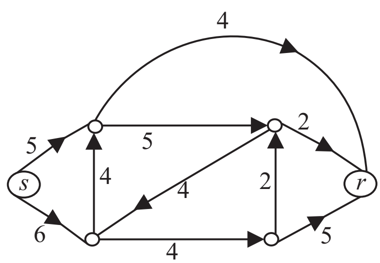



**Fig. 4.14**

$$[A, \bar{A}] = \{e \in E_D | e = uv, u \in A, v \notin A|\}.$$

$$f^+(A) = \sum_{e \in [\bar{A}, A]} f(e) \text{ and } f^-(A)$$

$$= \sum_{e \in [\bar{A}, A]} f(e).$$

Particularly,

$$f^+(u) = \sum_{v \in N} f(uv)$$

and $$f^-(u) = \sum_{v \in N} f(uv)$$

Hence

A flow in a network $N$ is a function

$f : V_N \times V_N \rightarrow IR$ such that $o \le f(e) \le A(e)\ \forall e$

and $f^-(v) = f^+(v)\ \forall v \notin \{s, r\}$

***Note***:

(*i*) *Every network N has a zero flow defined by* $f(e) = o \forall e$.

(*ii*) *For a flow of and each subset* $A \subseteq V_N$, *define the resultant flow from A and the value of f as the number* $val(f_A) = f^+(A) - f^-(A)$ *and* $val(f) = val(f_s) = f^+(s) - f^-(s)$

(*iii*) *A flow f of a network N is a maximum flow, if there does not exist any flow* $f'$ *such that* $val(f) < val(f')$

## 4.9 Improvable Flows

Let $f$ be a flow in a network $N$, and let $P = e_1 e_2 e_3 ... e_n$ be an undirected path in $N$ where an edge $e_i$ is along $p$, if $e_i = v_i v_{i+1} \in E_N$, and against $P$, if $e_i = v_{i+1} v_i \in E_N$

we define a non-negative number $l(p)$ for $P$ as follows:

$$l(P) = \min e_i\, l(e),$$

where $$l(e) = \{\alpha(e) - f(e)$$

for $e$ is along $f(e)$. for $e$ is againt $P\}$

we may define,

Let $f$ be a flow in a network $N$. A path $P : s \xrightarrow{*} r$ is ($f^-$) improvable, if $l(p) > 0$

On the right, the bold path has value $l(P) = 1$ and therefore this path is improvable. (see fig 4.15)

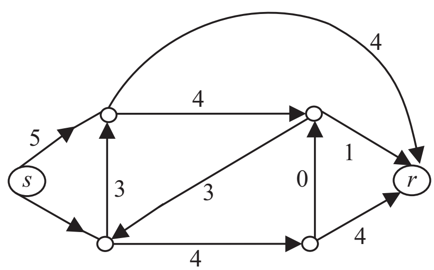



**Fig. 4.15**

### Lemma 4.1

Let $N = (D, s, r, \alpha)$ be a network with a flow $f$

(i) *If* $A \subseteq N \setminus \{s, r\}$, *then* $val(f_A) = 0$

(ii) $val(f) = -val(f_r)$.

Proof: Let $A \in N \setminus \{s, r\}$. Then

$$0 = \sum_{v \in A} (f^+(v) - f^-(v))$$

$$= \sum_{v \in A} (f^+(v) - \sum_{v \in A} f-(v)$$

$$= f^+(A) - f^-(A)$$

$$= \text{val}\ (f_A)$$

where the third equality holds since the values of the edge $uv$ with $u, v \in A$ cancel each out.

Similarly we can prove second claim.

### Lamma 4.2

*Let N be a network. If f be a maximum flow of N, then it has no improvable paths.*

***Proof*:**

We define,

$$f'(e) = \begin{cases} \{f(e) + l(p) \text{ if } e \text{ is along } P \\ f(e) - l(p) \text{ if } e \text{ is against } P \\ f(e) \text{ if } e \text{ is not in } P \end{cases}$$

Than $f'$ is a flow, since at each intermediate vertex $v \notin \{r, s\}$, we have $f(f')^-(v) = f(f')^+(v)$. and the capacities of the edges are not exceeded.

Now $\text{val}(f') = \text{val}\ (f) + l(p)$, since $p$ has exactly one edge $sv \in E_N$ for the source $s$. Hence, if $l(P) > 0$, then we can improve the flow (see Fig. 4.16)

■

## 4.10 Max-Flow Min-Cut Theorem

Let $N = (d, s, r, \alpha)$ be a network. For a subset $S \subset V_N$ with $s \in S$ and $r \notin S$, let the cut by $S$ be

$$[S] = [S, \overline{S}]\ (= \{uv \in E_N \mid u \in S, u \notin S\})$$

The capacity of the cut $[S]$ is the sum

$$\alpha[S] = \alpha^+(S) = \sum_{e \in [S]} \alpha(e)$$

A cut $[S]$ is a minimum cut, if there is no cut $[R]$ with $\alpha[R] < \alpha[S]$

for example in Fig. 4.17. The capacity of the cut for the indicate vertices is equal to 10.

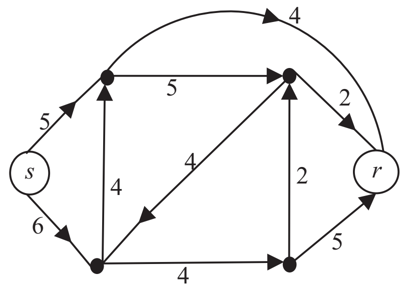



**Fig. 4.17**

## Lemma 4.3

For a flow $f$ and a cut $[S]$ of $N$, val$(f)$ = val $(f_s) = f^+(S) - f^-(S)$

***Proof:***

Let $S_i = S \setminus \{s\}$

Now, val $(S_i) = 0$ [since $S_i \subseteq N \setminus \{s, r\}$]

and val $(f)$ = val $(f_s)$

$$\text{val}(f_s) = \text{val}(f_s) - \sum_{v \in s_i} f(s,v) + \sum_{v \in s_i} f(v,s)$$

$$+ \text{val}(f_{s_i}) + \sum_{v \in s_i} f(sv) - \sum_{v \in s_i} f(vs)$$

$$= \text{val}(f_s) = \text{val}(f)$$

■

## Theorem 4.11

*For a flow $f$ and any cut $[S]$ of $N$, val$(f) \leq \alpha[S]$. Furthermore, equality holds if and only if for each $u \in S$ and $v \notin S$,*

($i$) if $e = uv \in E_N$, then $f(e) = a(e)$

($ii$) if $e = uv \in E_n$, then $f(e) = 0$.

## *Proof:*

We have from definition of flow,

$$f^+(S) = \sum_{e \in [s]} f(v) \leq \sum_{e \in [s]} \alpha(e) = a[S] \quad ...(i)$$

and $f^-(S) \geq 0$

Also val$(f)$ = val$(f_S) = f^+(S) - f^-(S)$

[Lemma 4.3]

Hence, $\text{val}(f) \leq \alpha[S]$, as required

Also, the equality $\text{val}(f) = \alpha[S]$ holds iff

(1) $f^{+}(S) = \alpha[S]$

and (2) $f^{-}(S) = 0$ ... $(ii)$

This holds iff $f(e) = \alpha(e)\ \forall\ e \in [S]$

$(\because f(e) \leq \alpha(e))$

and (2) $f(e) = 0\ \forall e = uv$

with $u \in S$ and $v \notin S$.

This proves the claim. ■

***Note*:**

*If f is a maximum flow and [S] a minimum cut, then val(f) ≤ α[S]*

## 4.11 *k*-flow

A $k$-flow $(H, \alpha)$ of an undirected graph $G$ is an orientation $H$ of $G$ together with an edge colouring $\alpha : E_H \to [0, k-1]$

such that for all vertices $u \in V$.

$$\sum_{f = uv \in E_H} \alpha(f) = \sum_{e = uv \in E_H} \alpha(e) \qquad \ldots (i)$$

*i.e.* The sum of the incoming values equals the sum of outgoing values. A $k$-flow is nowhere zero, if $\alpha(e) \neq 0\ \forall\ e \in E_H$.

In the $k$-flows we do not have any source or sink for our convenience,

let $\alpha(e^{-1}) = -\alpha(e)\ \forall\ e \in E_H$

in the orientation $H$ of $G$ so that the condition $(i)$ becomes

$$\sum_{e = uv \in E_H} \alpha(e) = 0 \qquad \ldots (ii)$$

The condition $(ii)$ generalizes to the subset $A \leq V_G$ in a natural way

$$\sum_{e \in [A, \bar{A}]} \alpha(e) = 0 \qquad \ldots (iii)$$

since the values of the edges inside $A$ cancel out each other.

***Note*:**

*If G has nowhere zero k-flow for some k, then G has no bridges.*

*for example, the graph fig 4.18 is with a nowhere zero 4-flow.*

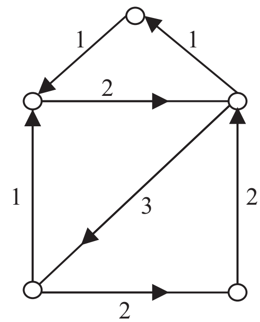



**Fig. 4.18**

## 4.12 Tutte's Problem

This problem was conjected by **TUTTE(1954)**. That every bridgeless graph has a nowhere zero 5-flow but does not have any nowhere 4-flows, and so 5 is the best one can think f. Tutte's conjecture resembles the 4-colour theorem, and indeed, the conjective is known to hold for the planar graphs. This proof uses the 4-colour theorem, which will be discussed later on.

### *Proposition 4.1:*

*A graph has a 2-flow if and only if all its degrees are even.*

### *Proof*:

Since a multigraph admits a $k$-flow iff it admits a $Z_k$-flow.

A graph $G = (V, E)$ has 2-flow iff it has a $Z_2$-flow *i.e.* iff the constant map.

$\vec{E} \to Z_2$ with value $\bar{1}$ satiesfies $(F_2)$.

This is the case iff all degrees are even. ■

### *Proposition 4.2:*

*A cubic graph has a 3-flow iff it is bipertite.*

### *Proof*:

Let $G = (V, E)$ be a cubic graph.

We assume that $G$ has a 3-flow, and hence a $Z_3$-flow $f$.

We can show that cycle $C = x_0x_1,\ldots, x_px_o$ in $G$ has been length.

We consider two consecutive edges on $C$

say $C_{i-1} : = x_{i\text{-}1.}$

If $f$ assigned the same value to these edges in the direction of the foreward orientation of $C$, *i.e.* $f(e_{i-1}, x_{i-1}, x_i) = f(e_i, x_i, x_{i+1})$, than $f$ could not satisfy $(F_2)$ at $x_i$ for any non-zero value of the third edge at $x_i$.

Therefore, $f$ assign the value $\bar{1}\,\bar{2}$ and to the edges of $C$ alternately, and in particular $C$ has even length.

Conversely, let $G$ is bipartite, with vertex bipartition $\{X, Y\}$. Since $G$ is cubic, the mapping $\vec{E} \to Z_3$ defined by

$f(e, x, y) := \bar{1}$ and $f(e, y, x) := \bar{2}$ for all edges $e = xy$ with $x \in X$ and $y \in Y$ is a $Z_3$-flow on $G$.

Hence $G$ has a 3-flow. ■

***Proposition 4.3:***

*For all even $n > 4$, $\phi(K^n) = 3$.*

***Proof*:**

$$\phi(K^n) \geq 3 \text{ for even } n.$$

By induction process on $n$, we show that every $G = K^n$ with even $n > 4$ has a 3-flow.

Let $n = 6$, there $G$ is the edge-disjoint union of three graphs $G_1, G_2, G_3$, with $G_1, G_2 = K^3$ and $G_3 = K_{3,3}$.

It is clear that $G_1$ and $G_2$ each have a 2-flow, whicle $G_3$ has a 3-flow. The union of all these flows is a 3-flow on $G$.

Now let $n > 6$, and assume the assertion holds for $n - 2$, clearly, $G$ is the edge-disjoint union of a $K^{n-2}$ and a graph $G' = (V', E')$ with $G' = K^{n-2} * K^2$.

The $K^{n-2}$ has a 3-flow by induction.

To find a $Z_3$-flow on $G'$.

For every vertex $z$ of the $K^{n-2} \subseteq G'$,

let $f_3$ be a $Z_3$-flow on the triangle $zxyz \subseteq G'$,

where $e = xy$ is the edge of the $K^2$ in $G'$.

Let $f$: $\vec{F}^1 \to Z_3$ be the sum of these flows.

It is clear that, $f$ is nowhere zero, except possible in $(e, x, y)$ and $(e, y, x)$.

If $f(e, x, y) \neq \bar{0}$ then $f$ is the desired $Z_3$-flow on $G'$. If $f(e, x, y) = 0$, then $f + f_z$ is a $Z_3$-flow on $G'$.

### *Proposition 4.4:*

*Every 4-edge-connected graphs has a 4-flow.*

### *Proof*:

Let $G$ be 4-edge connected graph. $G$ has two edge-disjoint spanning tree $T_i$, $i = 1, 2$.

For each edge $e \notin T_i$, let $C_{i,e}$ be the unique cycle in $T_{i+e}$ and let $f_{i,e}$ be a $\boldsymbol{Z}_4$-flow of value $\bar{i}$ around $C_{i,e}$ more precisely: a $\boldsymbol{Z}_4$-circulation on $G$ with values $\bar{i}$ and $-\bar{i}$ on the edges of $C_{i,e}$ and zero otherwise.

Let $$f_i := \sum_{e \in T_1} f_{i,e} \qquad \ldots(i)$$

since each $e \notin T_1$ lies only one cycle $C_1, e', f_1$ takes only the value $\bar{1}$ and $-\bar{1}$ $(= \bar{3})$ outside $T_1$.

Let $$F := \{e \in E(T_1) \mid f_1(\mathrm{e}) = \mathrm{o})\qquad \ldots(ii)$$

and $$f_2; = \sum_{e \in F} f_{2,e} \qquad \ldots(iii)$$

As above,

$$f_2(e) + \bar{2} = -\bar{2} \; \forall \; e \in F \qquad \ldots(iv)$$

Now, $f := f_1 + f_2$ is the sum of $Z_4$. Circulations, and hence itself a $Z_4$ circulation.

Moreover, $f$ is nowhere zero: on edges in $F$ it takes the value $\bar{2}$, on edges of $T_1$–$F$ it agrees with $f_1$, and on all edges outside $T_1$ it takes one of the values $\bar{1}$ or $\bar{3}$.

Hence $f$ is a $Z_4$-flow on $G$.

■

## SUMMARY

1. A **diagraph** $D = (V, E)$ consists of a finite set of vertices $V$ joined by a set of directed edges $E$. A directed edge from vertex $x$ to vertex $y$ is denoted by $(x, y)$.
2. The **indegree** of a vertex $v$, denoted by indeg $(v)$, is the number of edges terminating at $v$; its **outdegree**, denoted by outdeg $(v)$, is the number of edges leaving $v$. A vertex with indegree 0 is a **source**; a vertex with outdegree 0 is a **sink.**
3. The **adjacency matrix** of a diagraph is $A = (a_{ij})$, where $a_{ij}$ = number of directed edges from vertex $v_i$ to vertex $v_j$

4. Let $D$ be a diagraph with vertices $v_1, v_2, ..., v_n$ and $e$ edges. Then

$$\sum_{i=1}^{n} \text{indeg}(v_i) = e = \sum_{i=1}^{n} \text{outdeg}(v_i)$$

5. A loop-free digraph with exactly one edge between any two distinct vertics is a **dominance diagraph** or a **tournament.**

6. A **weighted digraph** has a weight $w$ for every edge.

7. The **weight** of a directed path is the sum of the weights of the edges along the path.

8. The **weighted adjacency matrix** of a digraph $(V, E)$ is $W = (w_{ij})$, where

$$\begin{cases} \infty & \text{if } (v_i, v_j) \notin E \\ \text{weight of edge } (i, j) & \text{otherwise} \end{cases}$$

9. A **shortest path** from $u$ to $v$ in a weighted digraph weighs the least.

10. Dijkstra's algorithm finds a shortest path and its length from the source to any vertex in a weighted digraph.

## EXERCISES

1. Show that $\sum_{v \in V} d^-(v) = \hat{\text{I}} = \sum_{v \in V} d^+(v)$.

2. Show that $D$ is diconnected if and only if $D$ is connected and each block of $D$ is disconnected.

3. Show that $G$ has an orientation $D$ s.t. $|1d^+(v) - d^-(\text{v})| \leq 1$ for all $v \in V$.

4. Show that if $D$ is strict and max $\{\delta^- \delta^+\} \setminus k > 0$, then $D$ contain a directed cycle of length at least $k + 1$.

5. Prove that every tournament has a directed Hamilton path.

6. Prove that $D(G)$ is $k$-are-connected iff $G$ is k-edge-connected.

7. Find a circular sequence of seven 0'S and seven 1's such that all 4-digit binary numbers except 0000 and 1111 appear as blocks of the sequence.

8. Show, by considering the Petersen graph, that the following statement is false: every graph $G$ has an orientation in which, for every $S \subseteq V$, the cordinalities of $(S, \overline{S})$ and $(\overline{S}, S)$ differ by atmost one.

**9.** Apply the method of ranking.

(*a*) the four tournaments shown in Fig. below.

(*b*) the tournament with adjacency matrix.

| | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| A | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 |
| B | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| C | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| D | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 |
| E | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| F | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| G | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 0 |
| H | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 |
| I | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| J | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 0 |

**10.** (*a*) Show that Nash-Williams' theorem is equivalent to the following statement: if every bond of $G$ has at least $2k$ edges, then there is an orientation of $G$ in which every bond has at least $k$ arcs in each direction.

(*b*) Show, by considering the Grotzsch graph, that the following analogue of Nash-Williams' theorem is false: if every cycle of $G$ has at least $2k$ edges, then there is an orientation of $G$ in which every cycle has at least $k$ arcs in each direction.

**11.** Let $v_1, v_2 ..., v_v$ be the vertices of a digraph $D$. The adjacency matrix of $D$ is the $v \times v$ matrix $A = [a_{ij}]$ in which $a_{ij}$ is the number of arcs of $D$ with tail $v_i$ and head $v_j$. Show that the $(i, j)$th entry of $A^k$ is the number of directed $(v_i, v_j)$-walks of length $k$ in $D$.

**12.** Let $D_1, D_2, ..., D_m$ be the components of $\hat{D}$. The condensation $\hat{D}$ of $D$ is a directed graph with $m$ vertices $w_1, w_2, ..., w_m$; there is an arc in $\hat{D}$ with tail $w_i$ and head $w_j$ if and only if there is an arc in $D$ with tail in $D_i$ and head in $D_j$. Show that the condensation $\hat{D}$ of D contains no directed cycle.

**13.** In a network $N$, Prove that the value val($f$) of a maximum flow equals the capacity a[$S$] of a minimum cut.

**14.** State ant prove seymour's 6-flows theorem.

**15.** Prove that a connected graph $G$ has a flow number $f(G) = 2$ if and only if it is eulerian.

## Suggested Readings


1. **J.A. Bondy** and **U.S.R. Murty**, *Graph Theory with Applications*, Elsevier, New York, 1976.
2. **T.H. Cormen**, *Introduction to Algorithms,* McGraw-Hill, New York, 1990.
3. **L. Ford** and **D. Fulkerson**, *Flows and Networks*, Prinecton University Press, Princeton, NJ, 1962.
4. **R.P. Girmaldi**, *Discrete and Combinatorial Mathematics*: *An Applied Introduction*, 4th ed., Addison-Wesley, Reading, MA, 1999, pp. 324-480.
5. **E. Horowitz** and **S.Sahni**, *Fundamentals of Data Structures,* Computer Science Press, Potomac, MD, 1976 pp. 301-334.


❑❑❑

# 5 Matching & Covering

Joseph Bernard Kruskal (1928–)

Joseph Bernard Kruskal (1928-) was born in New York city. He graduated from the university of Chicago in 1948 and received his Ph.D. from Princeton in 1954. He joined as an instructor at Princenton later on he became an assistant professor at the university of Wisconsin and shifted to the university of Michigan in 1958.
He joined Bell Telephone labs in 1959, a position he still holds. He has served as visiting professor at Yale, Columbia and Rutgers.
Kruskal is known for his algorithm in Graph theory which is used to solve so many problems of matching and covering.

## 5.1 Introduction

If we are given a graph and are asked to find in it as many independent edges as possible. How would we go about this? Will we be able to pair up all its vertices in this way? If not, how can we be sure that this is indeed impossible? Somewhat surprisingly, this basic problem does not only lie at the heart of numerous applications, it also gives rise to some rather interesting graph theory.

A subset $M$ of $E$ is called a matching in $G$, if its elements are linked and no two are adjacent in $G$; the two ends of an edge in $M$ are called matched under $M$. A matching $M$ saturates a vertex $v$, and $v$ is said to be $M$-saturated, if some edge of $M$ is incident with $v$; otherwise, $v$ is $M$-unsaturated. If every vertex of $G$ is $M$-saturated, the matching $M$ is perfect. $M$ is a maximum matching if $G$ has no matching $M'$ with $|M'|>|M|$;

It is clear that every perfect matching is maximum. Maximum and perfect matching in a graph are indicated in Fig. 5.1. ($a$ & $b$).


S. K. Yadav, *Advanced Graph Theory*,
https://doi.org/10.1007/978-3-031-22562-8_5

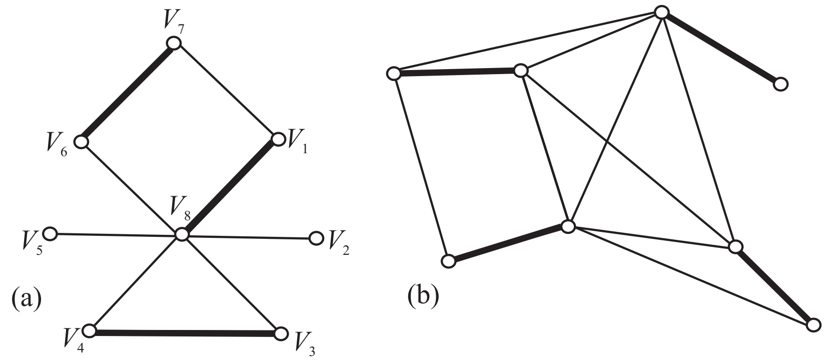



**Fig. 5.1: (a) A Maximum Matching; (b) a Prefect Matching**

Let $M$ be a matching in $G$. An M-alternating path in $G$ is a path whose edges are alternately in $E \setminus M$ and $M$. In Fig. 5.1($a$), the path $v_5\, v_8\, v_1\, v_7\, v_6$ in the graph is an $M$-alternating path.

An $M$-augmenting path is an $M$-alternating path whose origin and terminus are $M$-unsaturated.

Frankly speaking, a set $M$ of independent edges in a graph $G = (V, E)$ is called a matching. $M$ is a matching of $U \subseteq V$ if every vertex in $U$ is incident with an edge in $M$. The vertices in $U$ are then called matched (by $M$); vertices not incident with any edge of $M$ are unmatched.

A $k$-regular spanning subgraph is called a $k$-factor. Thus, a subgraph $H \subseteq G$ is a 1-factor of $G$ if and only if $E(H)$ is a matching of $V$. The problem of how to characterize the graphs that have a 1-factor, *i.e.* a matching of their entire vertex set, will be our main theme in this chapter.

Each vertex of $H$ has degree either one or two in $H$, since it can be incident with atmost one edge of $M$ and one edge of $M'$. Thus each component of $H$ is either an even cycle with edges alternately in $M$ and $M'$, or else a path with edges alternately in $M$ and $M'$. from equ.($i$), $H$ contains more edges of $M'$ than of $M$, and therefore some path component $P$ of $H$ must start and end with edges of $M'$. The origin and terminus of $P$, being $M'$-saturated in $H$, are $M$-unsaturated in $G$. Thus we can say that $P$ is an $M$-augmenting path in $G$.

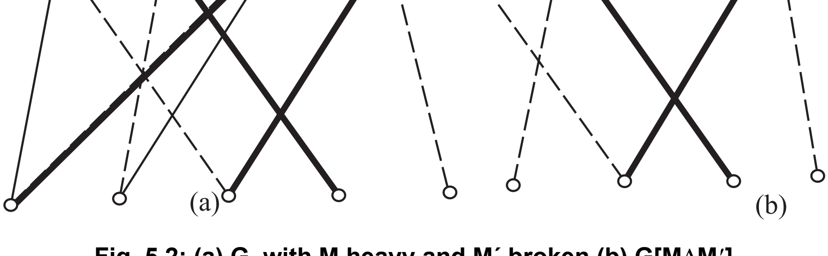



**Fig. 5.2: (a) G, with M heavy and M´ broken (b) G[M∆M′]**

***Theorem 5.1: (Berge Theorem, 1957)***

A matching $M$ in $G$ is a maximum matching, if $G$ contains no $M$-augmenting path.

***Proof:***

We consider a matching $M$ in a graph $G$, and suppose $G$ contains an $M$-augmenting path $v_0\, v_1\, v_2, \ldots, v_{2m+1}$

Redefining $M' \subseteq E$ by

$M' = (M \setminus \{v_1, v_2, v_3\, v_4, \ldots, v_{2m-1}, v_{2m}\} \cup \{v_0, v_1, v_2\, v_3, \ldots, v_{2m}\, v_{2m-1}\}$

Then $M'$ is a matching in $G$, and $|M'| = |M| + 1$

Thus $M$ is not a maximum matching.

Conversely, we consider that $M$ is not a maximum matching, and let $M'$ be a maximum matching in $G$.

Then, $$|M'| > |M| \qquad ...(i)$$

Set $H = G\,[M \,\Delta\, M']$, where $M \,\Delta\, M'$ denotes the symmetric difference of $M$ and $M'$. ■

## 5.2 Matching and Covering in Bipertite Graphs

For any set $S$ of vertices in $G$, we define the neighbour set of $S$ in $G$ to be the set of all vertices adjacent to vertices in $S$; this set is denoted by $N_G(S)$. Suppose, that $G$ is a bipartite graph with bipartition $(X, Y)$. In many applications one wishes to find a matching of $G$ that saturates very vertex in $X$; an example is the personnel assignment problem. Necessary and sufficient conditions for the existence of such a matching were first given by Hall (1935) (Theorem 5.3).

For this whole section, let $G = (V, E)$ be a fixed bipartite graph with bipartition $\{A, B\}$. Vertices denoted as $a$, $a'$ etc. will be assumed to lie in $A$, vertices denoted as $b$ etc. will lie in $B$.

How can we find a matching in $G$ with as many edges as possible? Let us start by considering an arbitrary matching $M$ in $G$. A path in $G$ which starts in $A$ at an unmatched vertex and then contains, alternately, edges from $E \setminus M$ and from $M$, is an alternating path with respect to $M$. An alternating path $P$ that ends in an unmatched vertex of $B$ is called an augmenting path (Fig. 5.3), because we can use it to turn $M$ into a larger matching: the symmetric difference of $M$ with $E(P)$ is again a matching (consider the edges at a given vertex), and the set of matched vertices is increased by two, the ends of $P$.

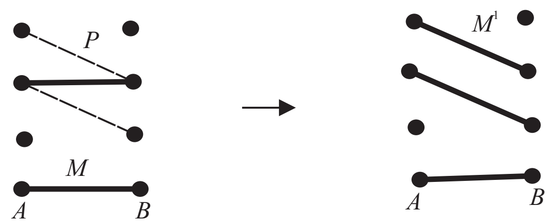



**Fig. 5.3: Augmenting the Matching *M* by the Alternating Path *P***

Alternating paths play and important role in the practical search for large matchings. In fact, if we start with any matching and keep applying augmenting paths until no further such improvement is possible, the matching obtained will always be an optimal one, a matching with the largest possible number of edges. The algorithmic problem of finding such matchings thus reduces to that of finding augmenting paths—which is an interesting and accessible algorithmic problem.

Our first theorem characterizes the maximal cardinality of a matching in $G$ by a kind of duality condition. Let us call a set $U \subseteq V$ a cover of $E$ (or a vertex cover of $G$) if every edge of $G$ is incident with a vertex in $U$.

### *Theorem 5.2: (König's Theorem 1931)*

*The maximum cardinality of matching in G is equal to the minimum cardinality of a vertex cover.*

***Proof:***

Let $M$ be a matching in $G$ of maximum cardinality. From every edge in $M$ let us choose one of its ends: its end in $B$ if some alternating path ends in that vertex, and its end in $A$ otherwise. We shall prove that the set $U$ of these $|M|$ vertices covers $G$; since any vertex cover of $G$ must cover $M$, there can be none with fewer than $|M|$ vertices, and so the theorem will follow.

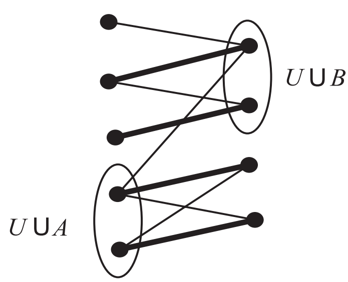



**Fig. 5.4 The vertex cover *U***

Let $ab \in E$ be an edge; we show that either $a$ or $b$ lies in $U$. If $ab \in M$, this holds by definition of $U$, so we assume that $ab \notin M$. Since $M$ is a maximal matching, it contains an edge $a'b'$ with $a = a'$ or $b = b'$. In fact, we may assume that $a = a'$; for if $a$ is unmatched (and $b = b'$), then $ab$ is an alternating

path, and so the end of $a'b' \in M$ chosen for $U$ was the vertex $b' = b$. Now if $a' = a$ is not in $U$, then $b' \in U$, and some alternating path $P$ ends in $b$: either $P' := P'$ (if $b \in P$) or $P' := Pb'a'b$. By the maximality of $M$, however, $P'$ is not an augmenting path. So $b$ must be matched, and was chosen for $U$ from the edge of $M$ containing it.

### *Theorem 5.3: [Hall, 1935] [Marriage Theorem]*

*G contains a matching of A if and only if* $|N(S)| \geq |S|$ *for all* $S \subseteq A$.

### *Proof*:

We give three proofs for the non-trivial implication of this theorem, *i.e.* that the 'marriage condition' implies the existence of a matching of $A$. The first of these is based on König's theorem; the second is a direct constructive proof by augmenting paths; the third will be an independent proof from first principles.

First proof. If $G$ contains no matching of $A$, then by Theorem 5.2 it has a cover $U$ consisting of fewer than $|A|$ vertices, say $U = A' \cup B'$ with $A' \subseteq A$ and $B' \subseteq B$. Then

$$|A'| + |B'| = |U| < |A|,$$

and hence

$$|B'| < |A| - |A'| = |A \setminus A'|$$

(Fig. 5.5). By definition of $U$, however, $G$ has no edges between $A \setminus A'$ and $B \setminus B'$, so

$$|N(A \setminus A') \leq |B'| < |A \setminus A'|$$

and the marriage condition fails for $S :\ A \setminus A'$.

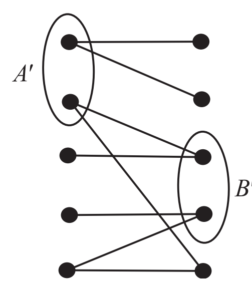



**Fig. 5.5: A Cover by Fewer than |*A*| Vertices**

***Second proof:*** Consider a matching $M$ of $G$ that leaves a vertex of $A$ unmatched; we shall construct an augmenting path with respect to $M$. Let $a_0$, $b_1$, $a_1$, $b_2$, $a_2$,…, be a maximal sequence of distinct vertices $a_i \in A$ and $b_i \in B$ satisfying the following conditions of all $i \geq 1$ (Fig. 5.6):

($i$) $a_0$ is unmatched;

($ii$) $b_i$ is adjacent to some vertex $a_{f(|i)} \in \{a_0, \ldots, a_{i-1}\}$;

($iii$) $a_ib_i \in M$.

By the marriage condition, our sequence cannot end in vertex of $A$: that $i$ vertices $a_0,\ldots,a_{i-1}$ together have a least $i$ neighbours in $B$, so we can always find a new vertex $b_i \neq b_1,\ldots, b_{i-1}$ that satisfies ($ii$). Let $b_k \in B$ be the last vertex of the sequence. By ($i$) – ($iii$),

$$P := b_k a_{f(k)} b_{f(k)} a_{f2(k)} a_{f3(k)}, \ldots, a_{fr}(k)$$

with $$f^r(k) = 0$$

is an alternating path.

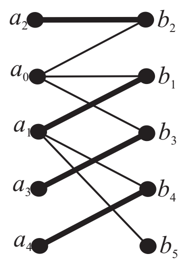



**Fig. 5.6: Proving the Marriage Theorem by Alternating Paths**

What is it that prevents us from extending our sequence further? If $b_k$ is matched, say to a, we can indeed extend it by setting $a_k := a$, unless $a = a_i$ with $0 < i < k$, in which case ($iii$) would imply $b_k = b_i$ with a contradiction. So $b_k$ is unmatched, and hence $P$ is an augmenting path between $a_0$ and $b_k$.

***Third proof:*** We apply induction on $|A|$. For $|A| = 1$ the assertion is true. Now let $|A| \geq 2$, and assume that the marriage condition is sufficient for the existence of a matching of $A$ when $|A|$ is smaller.

If $|N(S)| \geq |S| + 1$ for every non-empty set $S \subset A$, we pick an edge $ab \in G$ and consider the graph $G' := G - \{a, b\}$. Then every non-empty set $S \subseteq A \setminus \{a\}$ satisfies

$$|N_{G'}(S)| \geq |N_G(S) - 1| \geq |S|,$$

so by the induction hypothesis $G'$ contains a matching of $A \setminus \{a\}$. Together with the edge $ab$, this yields a matching of $A$ in $G$.

Suppose now that $A$ has a non-empty proper subset $A'$ with $|B'| = |A'|$ for $B' := N(A')$. By the induction hypothesis, $G' := G[A' \cup B']$ contains a matching of $A'$. But $G - G'$ satisfies the marriage condition too: for any set $S \subseteq A \setminus A'$ with $|N_{G-G'}(S) < |S|$ we would have $|N_G(S \cup A')| < |S \cup A'|$, contrary to our assumption. Again by induction, $G - G'$ contains a matching of $A \setminus A'$. Putting the two matchings together, we obtain a matching of $A$ in $G$.

■

### *Corollary 5.1:*

*If $|N(S)| \geq |S| - d$ for every set $S \subseteq A$ and some fixed $d \in N$, then G contains a matching of cardinality $|A| - d$.*

***Proof:***

We add $d$ new vertices to $B$, joining each of them too all the vertices in $A$. By the marriage theorem the new graph contains a matching of $A$, and at least $|A| - d$ edges in this matching must be edges of $G$. ■

### *Corollary 5.2:*

*If G is k-regular with $k \geq 1$, then G has a 1-factor.*

***Proof*:**

If $G$ is $k$-regular, then clearly $|A| = |B|$; it thus suffices to show that $G$ contains a matching of $A$. Now every set $S \subseteq A$ is joined to $N(S)$ by a total of $k|S|$ edges, and these are among the $k|N(S)|$ edges of $G$ incident with $N(S)$. Therefore $k|S| \leq k|N(S)|$, so $G$ does indeed satisfy the marriage condition. ■

### *Corollary 5.3:*

*If G is a k-regular bipartite graph with $k > 0$, then G has a perfect matching.*

***Proof*:**

Let $G$ be a $k$-regular bipartite graph with bipartition $(X, Y)$. Since $G$ is $k$-regular, $k|X| = |E| = k|Y|$ and so, since $k > 0$, $|X| = |Y|$. Now let $S$ be a subset of $X$ and denote by $E_1$ and $E_2$ the sets of edges incident with vertices in $S$ and $N(S)$, respectively. By definition of $N(S)$, $E_1 \subseteq E_2$ and therefore

$$k|N(S)| = |E_2| \geq |E_1| = k|S|$$

It follows that $|N(S) \geq |S|$ and hence, by theorem 5.3, that $G$ has a matching $M$ saturating every vertex in $X$. Since $|X| = |Y|$, $M$ is a perfect matching.

■

***Note*:**

*The above theorem is called marriage theorem since it can be more colourfully restated as follows: If every girl in a town knows exactly k boys, and every boy knows exactly k girls, then each girl can marry a boy she knows, and each boy can marry a girl he knows.*

### 5.2.1 *Covering*

A covering of a graph $G$ is a subset $k$ of $V$ such that every edge of $G$ has at least one and in $K$. A covering $K$ is a minimum covering if $G$ has no covering $K'$ with $|K'| < |K|$.

If $K$ is a covering of $G$, and $M$ is a matching of $G$, then $K$ contains at least one end of each of the edges in $M$. For any matching $M$ and any covering $K$, $|M| \leq |K|$.

If $M^*$ is a maximum matching and $\tilde{K}$ is a minimum covering, then

$$|M^*| \leq |\tilde{K}|$$

This result in due to König (1931) and closely related to Hall's Theorem. (see Fig. 5.7).

### *Lemma 5.1:*

*Let M be a matching and K be covering such that* $|M| = |K|$. *Then M is a maximum matching and K is a minimum covering.*

### *Proof*:

If $M^*$ is a maximum matching and $\tilde{K}$ is a minimum covering then,

$$|M| \leq |M^*| \leq |\tilde{K}| \leq |K|$$

since $\qquad |M| = |K|$

it follows that,

$$|M| = |M^*|$$

and $\qquad |K| = |\tilde{K}|$.

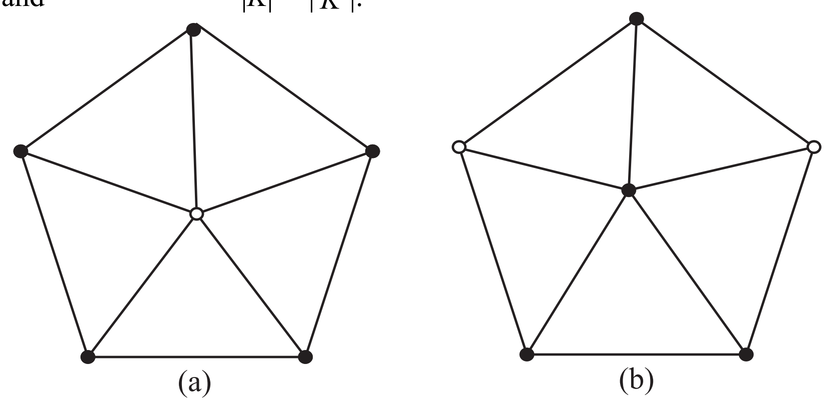



**Fig. 5.7 (a) A Covering; (b) a Minimum Covering**

### *Theorem 5.4:*

In a bipartite graph, the number of edges in a maximum matching is equal to the number of vertices in a minimum covering.

### *Proof*:

Let $G$ be a bipartite graph with bipartition $(X, Y)$, and let $M^*$ be a maximum matching of $G$. Denote by $U$ the set of $M^*$-unsaturated vertices in $X$, and by $Z$ the set of all vertices connected by $M^*$-alternating

paths to vertices of $U$. Set $S = Z \cap X$ and $T = Z \cap Y$. Then, we have that every vertex in $T$ is $M^*$-saturated and $N(S) = T$. Define $\tilde{K} = (X \setminus S) \cup T$ (see Fig. 5.7). Every edge of $G$ must have at least one of its ends in $\tilde{K}$. For, otherwise, there would be an edge with one end in $S$ and one end $Y \setminus T$, contradicting $N(S) = T$. Thus $\tilde{K}$ is a covering of $G$ and clearly

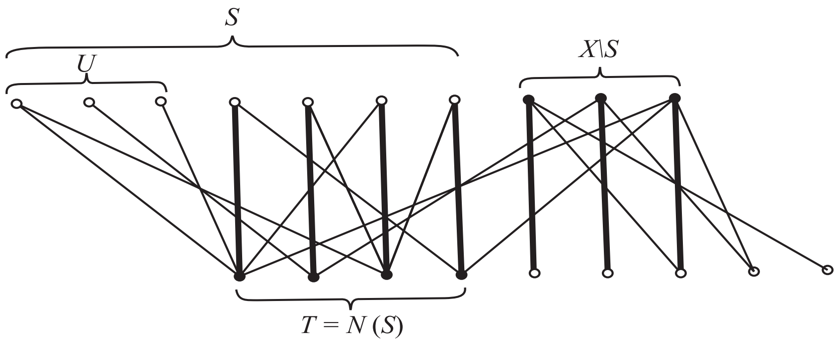



**Fig. 5.8**

By lemma 5.1, $\tilde{K}$ is a minimum covering, and the theorem follows.

■

## 5.3 Perfect Matching

A necessary and sufficient condition for a graph to have a perfect matching was obtained by Tutte (1947). A component of a graph is odd or even accordingly as it has an odd or even number of vertices.

Here $0(G)$ is the notation of number of odd components of $G$.

### *Theorem 5.5:*

*G has a perfect matching iff*

$0(G - S) \leq |S|$ for all $S \subseteq V$.

### *Proof*:

It is enough to prove this theorem for simple graphs.

Let $G$ has a perfect matching $M$.

Let $S$ be a proper subset of $V$.

Let $G_1, G_2, \ldots, G_n$ be the odd components of $G - S$.

Because $G_i$ is odd, some vertex $u_i$ of $G_i$ must be matched under $M$ with a vertex $v_i$ of $S$. (See Fig. 5.9)

$\therefore$ Since $\{v_1, v_2, \ldots, v_n\} \subseteq S$

$$0(G - S) = n = |\{v_1, v_2, \ldots, v_n\}| \leq |S|$$

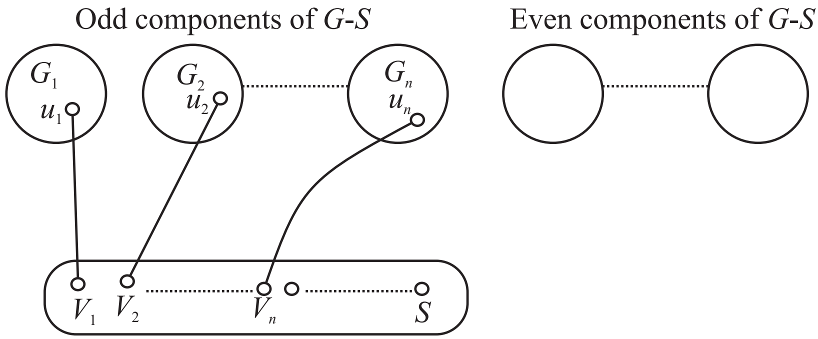



**Fig. 5.9 (a)**

Conversely,

Suppose that $G$ satisties $0(G-S) \leq |S|\ \forall\ S \subseteq V$ but has no perfect matching.

Let $G^*$ be a maximal graph having no perfect matching.

Then $G$ is a spanning subgraph of $G^*$.

Since $(G-S)$ is a spanning subgraph of $(G^*-S)$.

We have, $\quad 0(G^*-S) \leq 0(G-S)$

By assumption,

$$0(G^*-S) \leq 0(G-S) \leq |S|.$$

for all $\quad S \subseteq V(G^*)$.

In particular, setting $S = \phi$ we have,

$$0(G^*) = 0 \text{ and so } V(G^*) \text{ is even.}$$

Denote by $U$ the set of vertices of degree $(V-1)$ in $G^*$

If $\quad U = V$,

then $G^*$ has a perfect matching.

So we assume that $\quad U \neq V$.

***Claim*:**

$(G^*-U)$ is a disjoint union of complete graphs.

Suppose that some component of $(G^*-U)$ is not complete.

Then in this component, there are vertices $x, y, z \in G$

$xy \in E(G^*)$, $yz \in E(G^*)$ and $xz \notin E(G^*)$

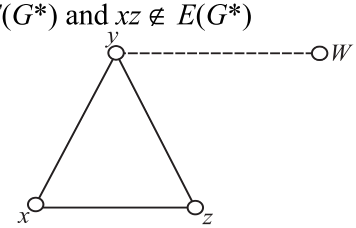



**Fig. 5.9 (b)**

Moreover, since $y \notin u$, there is a vertex $w$ in $(G^* - U)$ such that $yw \notin E(G^*)$.

Since G* is a maximal graph having no perfect matching,

$(G^* + e)$ has a perfect matching for all $e \notin E(G^*)$.

Let $M_1$ and $M_2$ be perfect matchings in $(G^* + xz)$ and $(G^* + yz)$ respectively.

Let $H$ be a subgraph of $G^*\ U\{xz, yw\}$ induced by $(M_1 \Delta M_2)$.

Since each vertex of $H$ has degree two, $H$ is a disjoint union of cycles.

Also, all of these cycles are even, since edges of $M_1$ alternate with edges of $M_2$ around them.

We distinguish two cases.

***Case 1:***

$xz$ and $yw$ are in different component of $H$.

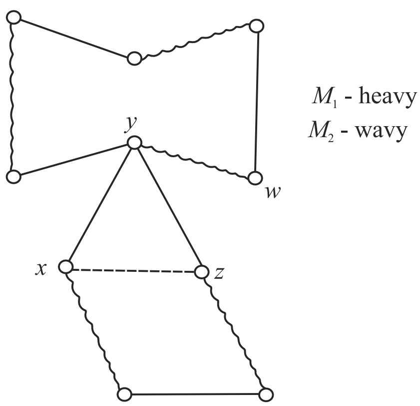



**Fig. 5.9 (c)**

Then if $yw$ is in the cycle $C$ of $H$, the edges of $M_1$ in $C$, together with the edges of $M_2$ not in $C$, constitute a perfect matching in $G^*$, contradicting the detinition of $G^*$.

***Case 2:***

$xz$ and $yw$ are in the same component of $H$.

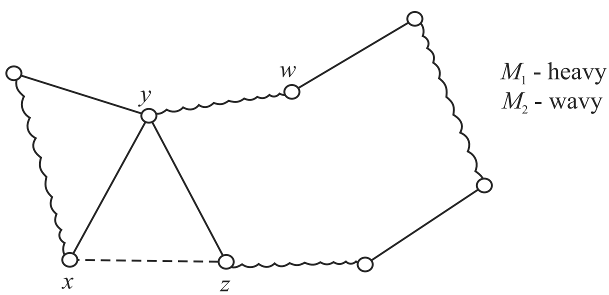



**Fig. 5.9(d)**

By symmetry of $x$ and $z$, we may assume that the vertices $x$, $y$, $w$ and $z$ occur in that order on $C$. (See Fig. 5.9 (d))

Then the edges of $M_1$ in the section $yw,\ldots, z$ of $c$ together with the edge $yz$ and the edges of $M_2$ not in the section $yw,\ldots, z$ of $c$, constitute a perfect matching in $G^*$, again contradicting the definition of $G^*$.

Since both case 1 and case 2 lead to contradictions it follows that $(G^* - U)$ is a disjoint union of complete graphs.

By case (1), $0(G^* - U) \leq |U|$

Thus atmost $|U|$ of the components of $(G^* - U)$ are odd.

But then $G^*$ clearly has a perfect matching.

One vertex in each odd component of $(G^* - U)$ is matched with a vertex of $U$; the remaining vertices in U, and in component of $(G^* - U)$ are then matched as shown in Fig. 5.10.

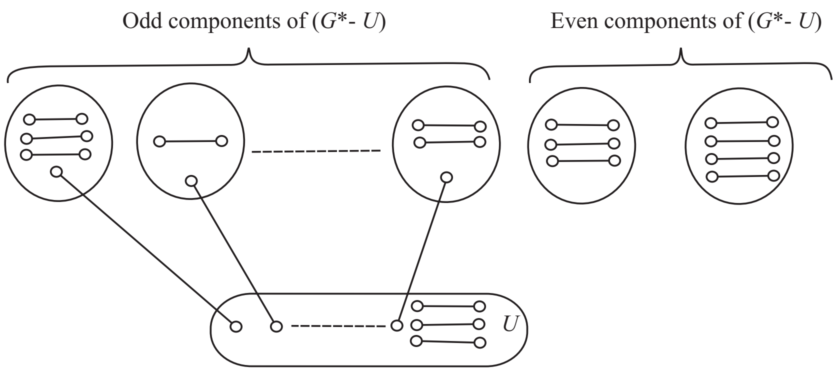



**Fig. 5.10**

Since $G^*$ was assumed to have no perfect matching. We obtained the desired contradiction.

Thus $G$ have a perfect matching.

■

### *Corollary 5.1:*

### *(Deduction of Petersen's Theorem from Tutte's Theorem)*

*Every 3-regular graph without cut edges has a perfect matching.*

### *Proof*:

Let $G$ be a 3-regular graph without cut edges and $S$ be a proper subset of $V$.

We denote by $G_1, G_2, G_3,\ldots, G_n$ the odd components of $(G - S)$.

Let $m_i$ be the number of edges with one end in $G_i$ and one end is $S$, $\forall 1 \leq i \leq n$.

Since $G$ is 3-regular

$$\sum_{v \in V(G_i)} d(v) = 3v\,(G_i)\ \forall 1 \le i \le n \qquad ...(i)$$

and

$$\sum_{v \in S} d(v) = 3|S| \qquad ...(ii)$$

from ($i$)

$$m_i = \sum_{v \in V(G_i)} d(v) - 2 \in (G_i) \text{ is odd}$$

Now $m_i \neq 1$ since $G$ has no cut edge.

Thus, $\quad m_i \ge 3$ for $1 \le i \le n \qquad ...(iii)$

It follows from ($ii$) & ($iii$) that,

$$0(G - S) = n \le \frac{1}{3}\sum_{i=1} m_i \le \frac{1}{3}\sum_{v \in S} d(v) = |S|$$

Hence $G$ has perfect matching.

***Note***:

if $G$ has a 1-factor, then clearly

$0(G - S) \le |S|$ for all $S \subseteq V|G|$,

since every odd component of $(G - S)$ will send a factor edge to $S$.

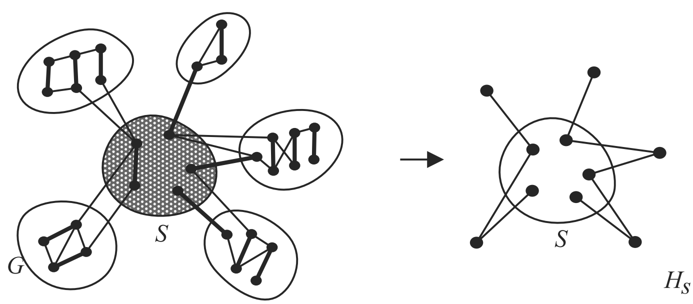



**Fig. 5.11: Deductive Graph**

## 5.4 Factor-critical Graph

A graph $G = (V, E)$ is called factor-critical if $G \neq \phi$ and $(\underline{G} - v)$ has a 1-factor for every vertex $v \in G$. Then $G$ itself has no a-factor because it has odd order.

We call a vertex set $S \subseteq V$ matchable to $(G - S)$ if the graph $H_S$, which arises from $G$ by contracting the components $C \in C_{G-S}$ to single vertices and deleting all the edges inside $S$, contains a matching of $S$. Formally $H_S$ is the graph with vertex set $S \cup C_{G-S}$ and edge set $\{sC \mid \exists\, c \in C : sc \in E\}$ (As shown in Fig. 5.11)

### *Theorem 5.6:*

*Every graph $G = (V, E)$ contains a vertex set $S$ with the following two properties:*

(*i*) *$S$ is matchable to $G - S$;*

(*ii*) *every component of $G - S$ is factor-critical.*

*Given any such set $S$, the graph $G$ contains a 1-factor if and only if $|S| = |C_{G-S}|$.*

### *Proof*:

For any given $G$, the assertion of Tutte's theorem follows easily from this result. Indeed, by (*i*) and (*ii*) we have $|S| \le |C_{G-S}| = 0(G - S)$ (since factor-critical graphs have odd order); thus Tutte's condition of $(G - S) \le |S|$ implies $|S| = |C_{G-S}|$, and the existence of a 1-factor follows from the last statement.

Note first that the last assertion of the theorem follows at once from the assertions (*i*) and (*ii*); if $G$ has a 1-factor, we have $0(G - S) \le |S|$ and hence $|S| = |C_{G-S}|$ as above;

conversely if $|S| = |C_{G-S}|$, then the existence of a 1-factor follows straight from (*i*) and (*ii*).

We now prove the existence of a set $S$ satisfying (*i*) and (*ii*). We apply induction on $|G|$. For $|G| = 0$ we may take $S = \emptyset$. Now let $G$ be given with $|G| > 0$, and assume the assertion holds for graphs with fewer vertices.

Let $d$ be the least non-negative integer such that

$$0(G - T) \le |T| + d \text{ for every } T \subseteq V. \qquad \text{... } (i)$$

Then there exists a set $T$ for which equality holds in (1); this follows from the minimality of $d$ if $d > 0$, and from $0(G - \emptyset) \ge |\emptyset| + 0$ if $d = 0$.

Let $S$ be such a set $T$ of maximum cardinality, and let $C := C_{G-S}$.

We first show that every component $C \in C$ is odd. If $|C|$ is even, pick a vertex $c \in C$, and let $S' := S \cup \{c\}$ and $C' := C - c$. Then $C'$ has odd order, and thus has at least one odd component. Hence, $0(G - S') \ge 0(G - S) + 1$. Since $T := S$ satisfies (1) with equality, we obtain

$$q(G - S') \ge q(G - S) + 1 = |S| \mid d + 1 = |S'| + d \ge q(G - S') \qquad \text{... } (ii)$$

with equality, which contradicts the maximality of $S$.

Next we prove the assertion (*ii*), that every $C \in C$ is factor-critical. Suppose there exist $C \in C$ and $c \in C$ such that $C' := C - c$ has no 1-factor. By the

induction hypothesis (and the fact that, as shown earlier, for fixed $G$ our theorem implies Tutte's theorem) there exists a set $T' \subseteq V(C')$ with

$$q(C' - T') > |T'|.$$

Since $|C|$ is odd and hence $|C'|$ is even, the numbers $0(C' - T')$ and $[T']$ are either both even or both odd, so they cannot differ by exactly 1. We may therefore sharpen the above inequality to

$$0(C' - T') \geq [T'] + 2.$$

For $T := S \cup \{c\} \cup T'$ we thus obtain

$$\begin{aligned} 0(G - T) &= 0(G - S) - 1 + 0(C' - T') \\ &\geq |S| + d - 1 + |T'| + 2 \\ &= |T| + d \\ &\geq 0(G - T) \end{aligned}$$

with equality, again contradicting the maximality of $S$.

It remains to show that $S$ is matchable to $G - S$. If $S = \emptyset$, this is trivial, so we assume that $S \neq \emptyset$. Since $T := S$ satisfies (1) with equality, this implies that $C$ too is non-empty. $H := H_S$, but 'backwards', *i.e.* with $A := C$. Given $C' \subseteq C$, set $S' := N_H(C') \subseteq S$. Since every $C \in C'$ is an odd component also of $G - S'$, we have

$$|N_H(C')| = |S'| \geq 0(G - S) - d \geq |C'| - d.$$

$H$ contains a matching of cardinality

$$|C| - d = 0(G - S) - d = |S|,$$

which is therefore a matching of $S$. ■

## 5.5 Complete Matching

A complete matching of a vertex set $V_1$ into those in $V_2$ of the bipertite graph with partition $V_1$ and $V_2$ is a matching of the graph in which there is one edge incident with every vertex in $V_1$ .*i.e.* every vertex of $V_1$ is matched against one vertex is $V_2$.

The complete matching need not to exist for the entire bipertite graphs. We can observe the fact by considering the interview with large number of suitable candidates for less number of position.

### *Theorem 5.7:*

*A complete matching of the vertex set $V_1$ and $V_2$ in a bipertite graph exists if and only if every subset of r vertices in $V_1$ is collectively adjacent to r or more vertices in $V_2$ for all values of r.*

***Proof***:

As a homework. (see problems)

***Problems 5.1:***

*Find whether a complete matching of $V_1$ into $V_2$ exists in the graph* (Fig. 5.12)

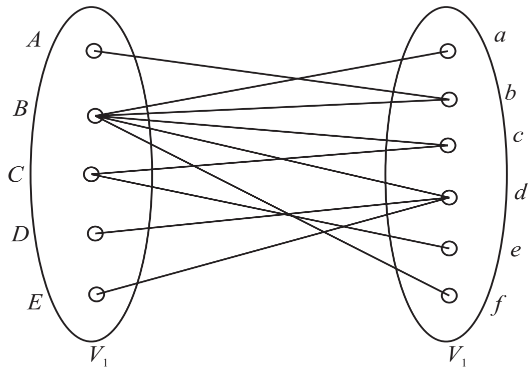



**Fig. 5.12**

***Solution***:

No, Because we have to take subset $\{D, E\}$ of $V_1$ having two vertices, then the elements of this set is collectively adjacent to only the subset $\{d\}$ of $V_2$. The cardinality of $\{d\}$ is one that is less than the cardinality of the set $\{D, E\}$.

***Problem 5.2:***

*Find whether a complete matching of $V_1$ into $V_2$ exist for the following graph? What can you say from $V_2$ to $V_1$*. (see Fig. 5.13)

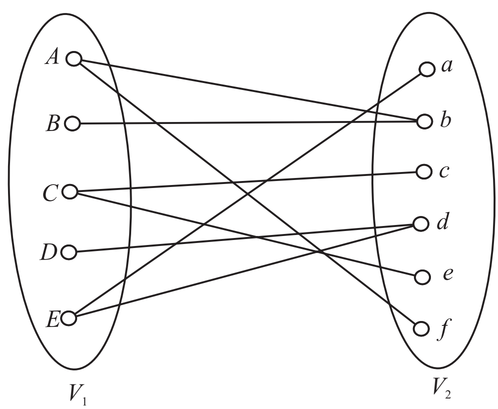



**Fig. 5.13**

***Solution***:

Yes, we get a complete matching from $V_1$ into $V_2$ which is $\{Af, Bb, Cc, Dd, Ea\}$

This matching is not unique, because

$\{Af, Bb, Cc, Dd, Ea\}$ is also a complete matching from $V_1$ into $V_2$.

Complete matching from $V_2$ into $V_1$ does not exist because cardinality of $V_2$ is more than the cardinality of $V_1$. ■

***Theorem 5.8:***

*For any bipertite graph G with partition $V_1$ and $V_2$, if these exists a positive integer M satisfying the condition that $deg_G(V_1) \geq M \geq deg_G(V_2)$, for all vertices $v_1 \in V_1$, $v_2 \in V_2$ then a complete matching of $V_1$ into $V_2$ exists.*

***Proof***:

Let $G$ be a bipertite graph with partition $V_1$ and $V_2$.

Let $M$ be a positive integer satisfying the condition that $\deg_G(v_1) \geq M \geq \deg_G(v_2)$, for all vertices $v_1 \in V_1$ & $v_2 \in V_2$ consider a r-element subset $S$ of the set $V_1$.

Since $\deg(r_1) \geq M$ from each element of $S$, there are atleast $M$ edges incident to the vertices in $V_1$. (since no vertex of $V_1$ is adjacent to the vertex of $V_2$.

Thus there are $M.\ r$ edges incident from the set $S$ to the vertices in $V_1$, but degree of every vertex of $V_2$ cannot exceed $M$ implies that these $M$.r edges are incident on atleast

$$\frac{M.r}{M} = \text{r vertices in } V_2$$

Hence, there exists a complete matching of $V_1$ *into* $V_2$. ■

## 5.6 Matrix Method to Find Matching of a Bipertite Graph

The adjacency matrix of the bipartite graph $G$ can be written by the rearrangement of rows and columns as

$$X(G) = \begin{bmatrix} 0 & X_{n_1 \times n_2} \\ X^T{}_{n_1 \times n_2} & 0 \end{bmatrix}$$

where $X_{n_1 \times n_2}$ is (0, 1) whose rows are corresponds the vertices of $V_1$ and columns corresponds to the vertices of $V_2$. A matching $V_1$ into $V_2$ such that no row or a column has more than one 1. The matching is complete if the $n_1 \times n_2$ matrix so obtained contains exactly one in each row.

### *Problem 5.3:*

*Find a complete matching of the graph* (Fig. 5.14) *by matrix method.*

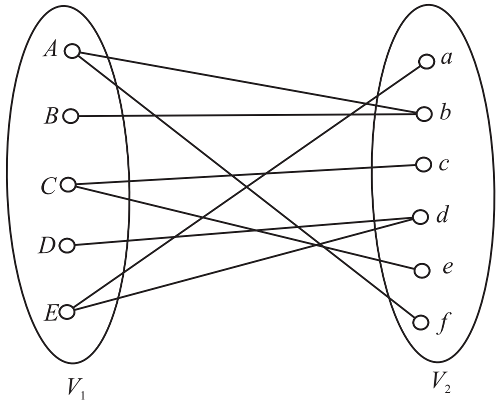



**Fig. 5.14**

### *Solution*:

$$X(G) = \begin{bmatrix} 0 & X_{n_1 \times n_2} \\ X^T{}_{n_1 \times n_2} & 0 \end{bmatrix}$$

Where $X n1 \times n2 =$

$$\begin{array}{c|cccccc|} & a & b & c & d & e & f \\ A & 0 & 1 & 0 & 0 & 0 & 1 \\ B & 0 & 1 & 0 & 0 & 0 & 0 \\ C & 0 & 0 & 1 & 0 & 1 & 0 \\ D & 0 & 0 & 0 & 1 & 0 & 0 \\ E & 1 & 0 & 0 & 1 & 0 & 0 \end{array}$$

here $n_1 = 5$, $n_2 = 6$ & $n = n_1 + n_2 = 11$ = total numbers of vertices of $G$.

***Step 1***: Choose the row $B$ and the column $b$.
(since $B$ contains 1 in only one place in the entire now)

***Step 2***: Discard the column $b$ (since it has already chosen)

***Step 3:*** Choose the row $D$ and the column $d$ (Since $D$ contains 1 in only one place in the entire row)

***Step 4:*** Discard the column $d$ (Since it has already chosen)

***Step 5:*** Choose the row $E$ and the column a (Since $Ea^{\text{th}}$ entry is one and not chosen earlier)

***Step 6:*** Discard the column $a$ (chosen in Step 5)

***Step 7:*** Choose the row $A$ and column $f$ (since the row $A$ contains exactly one 1 in the column $f$).

***Step 8:*** Discard the column $f$ (chosen in step 7)

***Step 9:*** Choose the row $C$ and the column $e$ (or $c$) [since $C$ is the final (unique) row]

***Step 10:*** Now, no row is left to choose and all the rows are able to be chosen. Hence the matching is complete (if any zero row remains, then the matching is not complete but the process slops)

The resultant matrices after each step and the final matching is given below.

| After the steps 1 and 2 | After the steps 3 and 4 | After the steps 5 and 6 | After the steps 7 and 8 |
|---|---|---|---|
| $\begin{array}{c} \\ A \\ C \\ D \\ E \end{array}\begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \left|\begin{array}{ccccc} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{array}\right| \end{array}$ | $\begin{array}{c} \\ A \\ C \\ E \end{array}\begin{array}{c} \begin{array}{cccc} a & c & e & f \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$ | $\begin{array}{c} \\ A \\ C \end{array}\begin{array}{c} \begin{array}{ccc} c & e & f \end{array} \\ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \end{array}$ | $\begin{array}{c} \\ C \end{array}\begin{array}{c} \begin{array}{cc} c & e \end{array} \\ \begin{pmatrix} 1 & 1 \end{pmatrix} \end{array}$ |

Resultant matrix and the corresponding matching are shown in Fig. 5.15

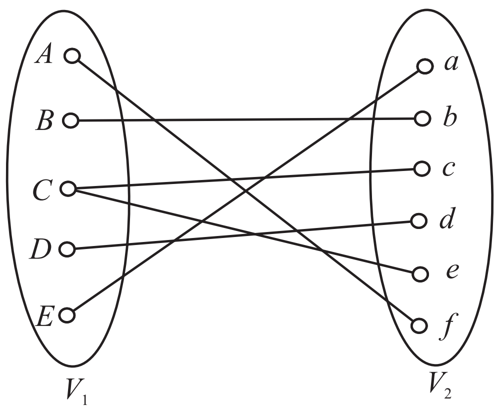



**Fig. 5.15**

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

The complete matching is {Af, Bb, Ce, Dd, Ea}

## 5.7 Path Covers

In this section we put the above question more generally: how many paths in a given directed graph will suffice to cover its entire vertex set? Of course, this could be asked just as well for undirected graphs. As it turns out, however, the result we shall prove is rather more trivial in the undirected case (exercise), and the directed case will also have an interesting corollary.

A directed path is a directed graph $P \neq \emptyset$ with distinct vertices $x_0, \ldots, x_k$ and edges $e_0, \ldots, e_{k-1}$ such that $e_i$ is an edge directed from $x_i$ to $x_{i+1}$, for all $i < k$. We denote the last vertex $x_k$ of $P$ by ter($P$). In this section, path will always mean 'directed path'. A path cover of a directed graph $G$ is a set of disjoint paths in $G$ which together contain all the vertices of $G$. Let us denote the maximum cardinality of an independent set of vertices in $G$ by $\alpha(G)$.

### *Theorem 5.9: (Gallai & Milgram Theorem, 1960)*

*Every directed graph G has a path cover by at most $\alpha(G)$ paths*

### *Proof:*

Given two path covers $p_1$, $p_2$ of a graph, we write $p_1 < p_2$ if $\{\text{ter}(P) \mid P \in p_1\} \subseteq \{\text{ter}(P) \mid P \in p_2\}$ and $|p| < |p|_2$. We shall prove the following:

If $p$ is a <-minimal path cover of $G$, then $G$ contains an independent set $\{v_P \mid P \in p\}$ of vertices with $v_P \in P$ for every $P \in p$.

Clearly, ($i$) implies the assertion of the theorem.

We prove ($i$) by induction on $|G|$. Let $p = \{P_1, \ldots, P_m\}$ be given as in ($i$), and let $v_i := \text{ter}(P_i)$ for every $i$. If $\{v_i \mid 1 \leq i \leq m\}$ is independent, there is nothing more to show; we may therefore assume that $G$ has an edge from $v_2$ to $v_1$. Since $P_2v_2v_1$ is again a path, the minimality of $p$ implies that $v_1$ is not the only vertex of $P_1$; let $v$ be the vertex preceding $v_1$ on $P_1$. Then $p' := \{P_1v, P_2, \ldots, P_m)$ is a path cover of $G' := G - v_1$. We first show that $p'$ is <-minimal with this property.

Suppose that $p'' < p'$ is another path cover of $G'$. If a path $P \in p''$ ends in $v$, we may replace $P$ in $p''$ by $Pvv_1$ to obtain a smaller path cover of $G$ than $p$, a contradiction to the minimality of $p$. If a path $P \in p''$ ends in $v_2$ (but none in $v$), we replace $P$ in $p''$ by $Pv_2v_1$, again contradicting the minimality of $p$. Hence $\{\text{ter}(P) \mid P \in p''\} \subseteq \{v_3, \ldots, v_m\}$, and in particular $|p''| \leq |p| - 2$. But now $p''$ and the trivial path $\{v_1\}$ together form a path cover of $G$ that contradicts the minimality of $p$.

Hence $p'$ is minimal, as claimed. By the induction hypothesis, $\{V(P) \mid P \in p'\}$ has an independent set of representatives. But this is also a set of representatives of $p$, and ($i$) is proved. ■

## 5.8 Applications

### 5.8.1 *The Personnel Assignment Problem*

In a certain company, there are $n$ worker $X_1, X_2, X_3, \ldots, X_n$, available for $n$ jobs $Y_1, Y_2, Y_3, \ldots, Y_n$. Each worker being qualified for one or more of these jobs. The questions arise; can all the men be assigned, one man per job, to jobs for which they are qualified? This problem is called the personnel assignment problem.

We construct a bipartite graph $G$ with bipartition $(X, Y)$, where $X = \{x_1, x_2, \ldots, x_n), Y = \{y_1, y_2, \ldots, y_n)$ and $x_i$ is joined to $y_i$ if and only if worker $X_i$ is qualified for job $Y_j$. The problem becomes one of determining whether or not $G$ has a perfect matching. According to Hall's theorem either $G$ has such a matching or there is a subset $S$ of $X$ such that $|N(S) < |S|$. Given any bipartite graph $G$ with bipartition $(X, Y)$, the algorithm either finds a matching of $G$ that saturates every vertex in $X$ or, failing this, finds a subset $S$ of $X$ such that $|N(S)| < |S|$.

We start with an arbitrary matching $M$. If $M$ saturates every vertex in $X$, then it is a matching of the required type. If not, we choose an $M$-unsaturated vertex $u$ in $X$ and systematically search for an $M$-augmenting path with origin $u$. Our method of search, to be described in detail below, finds such a path $P$ if one exists; in this case $\hat{M} = M \Delta E(P)$ is a larger matching than $M$, and hence saturates more vertices in $X$. We then repeat the procedure with $\hat{M}$ instead of $M$. If such a path does not exist, the set $Z$ of all vertices which are connected to $u$ by $M$-alternating paths is found. Then $S = Z \cap X$ satisfies $|N(S)| < |S|$.

Let $M$ be a matching in $G$, and let $u$ be an $M$-unsaturated vertex in $X$. A tree $H \subseteq G$ is called an $M$-alternating tree rooted at $u$ if (*i*) $u \in V(H)$, and (*ii*) for every vertex $v$ of $H$, the unique $(u, v)$-path in $H$ is an $M$-alternating path. An $M$-alternating tree in a graph is shown in Fig. 5.17.

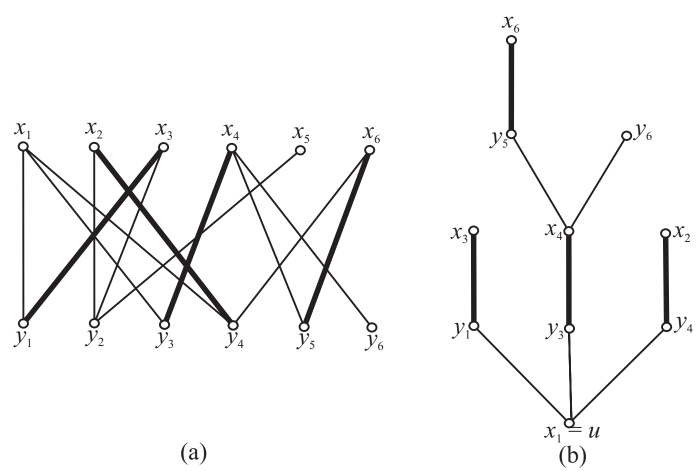



**Fig. 5.17: (a) A Matching M in *G*; (b) an M-alternating Tree in *G***

The search for an $M$-augmenting path with origin $u$ involves 'growing' an $M$-alternating tree $H$ rooted at $u$. Initially, $H$ consists of just the single vertex $u$. It is then grown in such a way that, at any stage, either

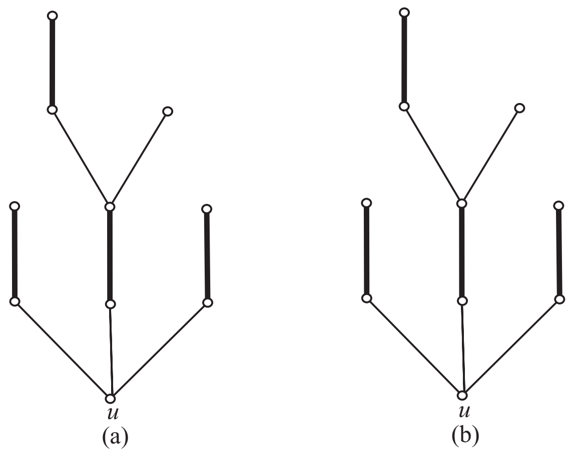



**Fig. 5.18: (a) Case (*i*) (b) Case (*ii*)**

($i$) all vertices of $H$ except $u$ are $M$-saturated and matched under $M$ (as Fig. 5.18a), or

($ii$) $H$ contains an $M$-unsaturated vertex different from $u$ (as in Fig. 5.18 b).

If ($i$) is the case (as it is initially) then, setting $S = V(H) \cap X$ and $T = V(H) \cap Y$, we have $N(S) \supseteq T$; thus either $N(S) = T$ or $N(S) \supset T$.

($a$) If $N(S) = T$ then, since the vertices in $S \backslash \{u\}$ are matched with the vertices in $T$, $|N(S) = |S| - 1$, indicating that $G$ has no matching saturating all vertices in $X$.

($b$) If $N(S) \supset T$, there is a vertex $y$ in $Y \setminus T$ adjacent to a vertex $x$ in $S$. Since all vertices of $H$ except $u$ are matched under $M$, either $x = u$ or else $x$ is matched with a vertex of $H$. Therefore $xy \notin M$. If $y$ is $M$-saturated, with $yz \in M$, we grow $H$ by adding the vertices $y$ and $z$ and the edges $xy$ and $yz$. We are then back in case($i$). If $y$ is $M$-unsaturated, we grow $H$ by adding the vertex $y$ and the edge $xy$, resulting in case ($ii$). The ($u$, $y$)-path of $H$ is then an $M$-augmenting path with origin $u$, as required.

Figure 5.19 illustrates the above tree-growing procedure.

The algorithm described above is known as the Hungarian method, and can be summarised as follows:

We start with an arbitrary matching $M$.

1. If $M$ saturates every vertex in $X$, stop. Otherwise, let $u$ be an $M$-unsaturated vertex in $X$. Set $S = \{u\}$ and $T = \emptyset$.

2. If $N(S) = T$ then $|N(S)|<|S|$, since $|T| = |S| - 1$. Stop, since by Hall's theorem there is no matching that saturates every vertex in $X$. Otherwise, let $y \in N(S) \backslash T$.
3. If $y$ is $M$-saturated, let $yz \in M$. Replace $S$ by $S \cup \{z\}$ and $T$ by $T \cup \{y\}$ and go to step 2. (Observe that $|T| = |S| - 1$ is maintained after this replacement.) Otherwise, let $P$ be an $M$-augmenting $(u, y)$-path. Replace $M$ by $\hat{M} = M$ D $E(P)$ and go to step 1.

Consider, for example, the graph $G$ in Fig. 5.20a, with initial matching $M = \{x_2y_2, x_3y_3, x_5y_5\}$. In Fig.5.20 b an $M$-alternating tree is grown, starting with $x_1$, and the $M$-augmenting path $x_1y_2x_2y_1$ found. This results in a new matching $\hat{M} = (x_1y_2, x_2y_1, x_3y_3, x_5y_5)$, and an $\hat{M}$ -alternating tree is now grown from $x_4$ (figures 5.20 c and 5.20 d) Since there is no $\hat{M}$ -augmenting path with origin $x_4$, the algorithm terminates. The set $S = \{x_1, x_3, x_4)$, with neighbour set $N(S) = (y_2, y_3)$, shows that $G$ has no perfect matching.

A flow diagram of the Hungarian method is given in Fig. 5.21. Since the algorithm can cycle through the tree-growing procedure, I, at most $|X|$ times before finding either an $S \subseteq X$ such that $|N(S)| < |S|$ or an $M$-augmenting path, and since the initial matching can be augmented at most $|X|$ times before a matching of the required type is found.

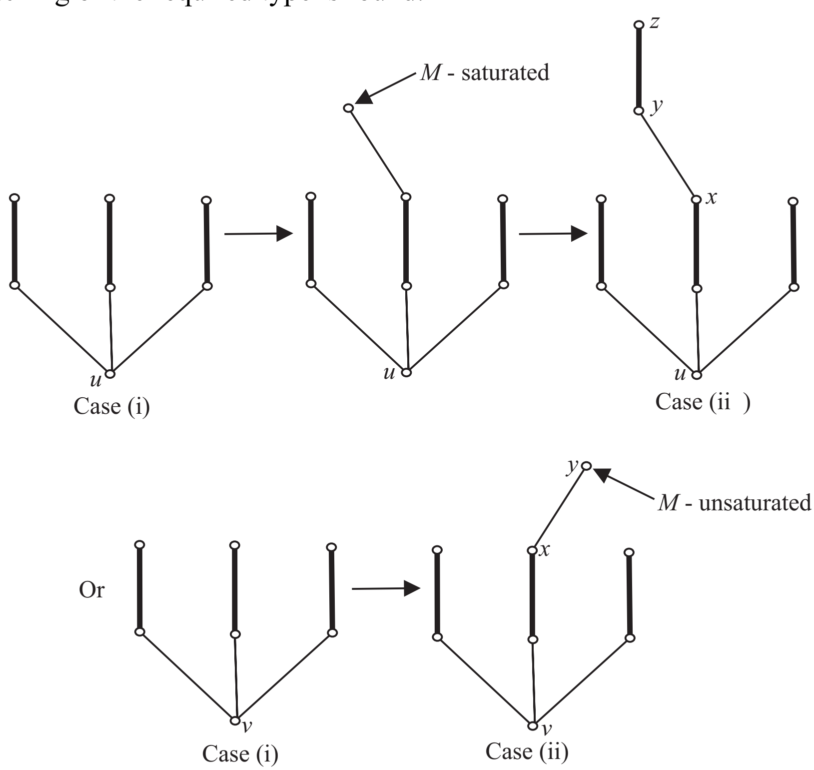



**Fig. 5.19: The tree-growing Procedure**

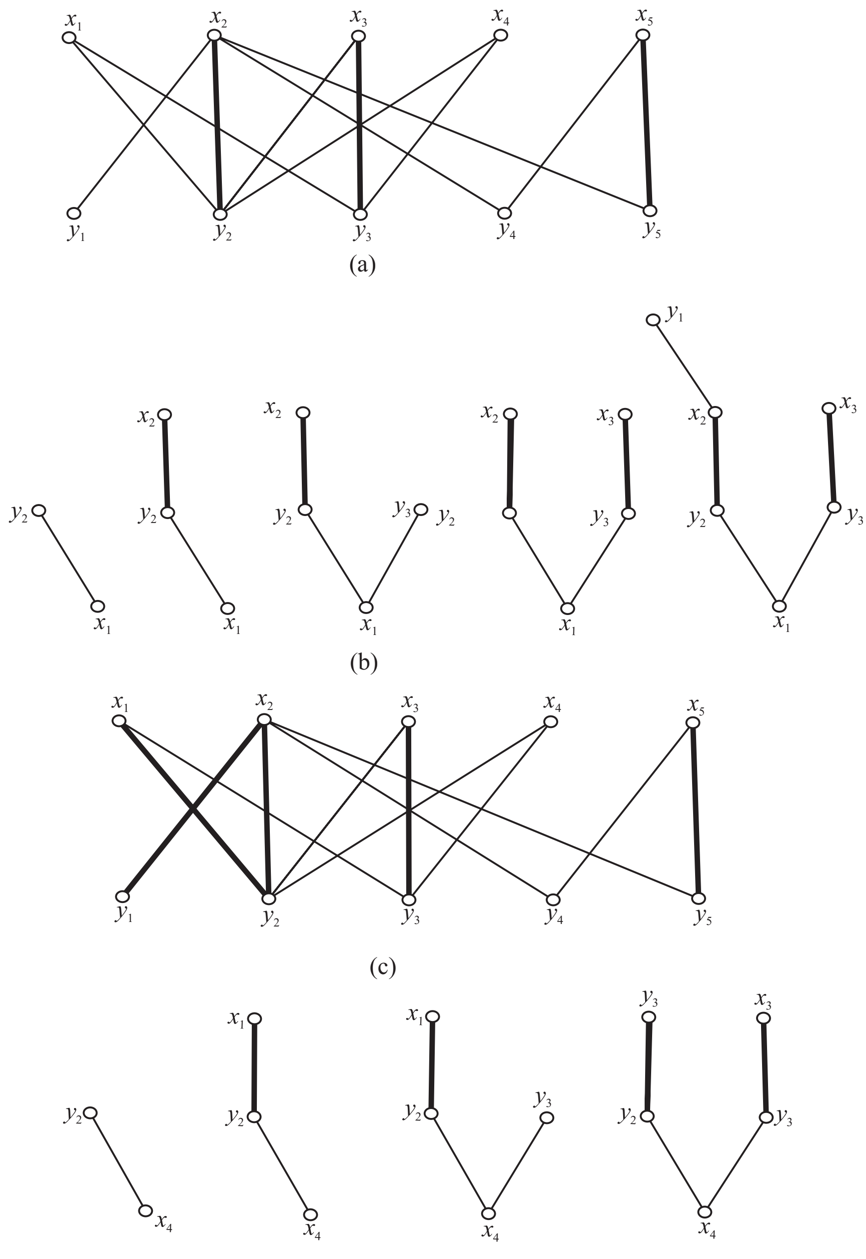



**Fig. 5.20: (a) Matching *M* ; (b) an *M*-alternating tree; (c) Matching *M*; (d) an *M*-alternating tree**

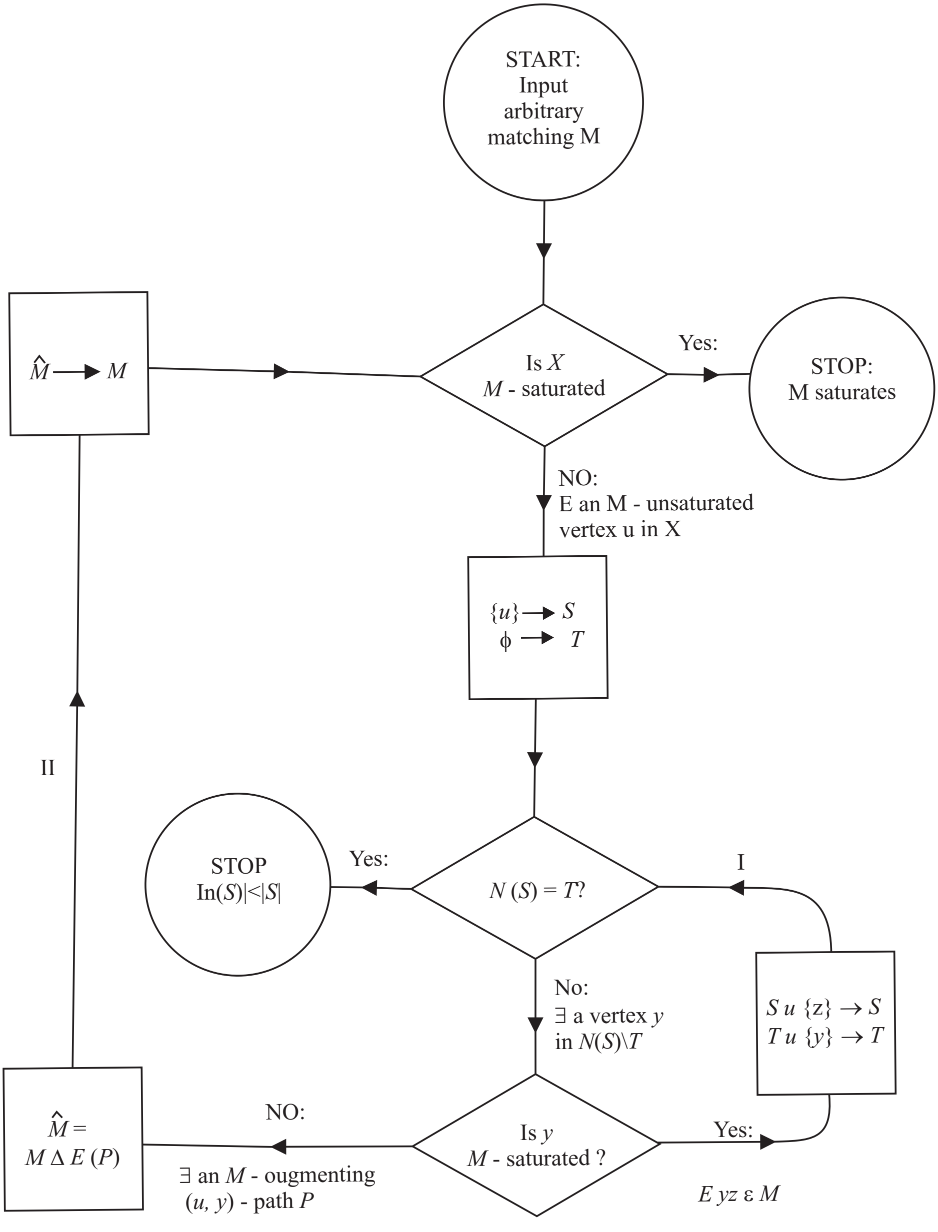



**Fig. 5.21 Flow Chart**

## 5.8.2 *The Optimal Assignment Problem*

The Hungariam method is the efficient way to determine a feasible assignment of workers to jobs, if one exists. If we wish to take into account the effectiveness of the workers in their various jobs. On this care we are interested in an assignment that maximises the total effectiveness of the workers. This problem of finding such an assignment is called the optimal assignment problem.

We consider a weighted complete bipertite graph with bipartition $(X, Y)$, where $X = \{x_1, x_2, x_3,\ldots, x_n\}$, $Y = \{y_1, y_2, y_3,\ldots, y_n)$, and edge $x_i\, y_j$ has weight $w_{ij} = w(x_i.y_j)$, the effectiveness of workers $X_i$ in job $Y_i$. The optimal assignment problem in clearly equivalent to that of finding a maximum-weight perfect matching in this weighted graph.

We define a feasible vertex labelling as a real-valued function $l$ on the vertex set $X \cup Y$ such that, for all $x \in X$ and $y \in Y$

$$l(x) + l(y) \geq w(xy) \qquad \ldots(i)$$

(*The real number l(v) is called the lable of the vertex v.*)

A feasible vertex labelling is thus a labelling of the vertices such that the sum of the labels of the two ends of an edge is at least as large as the weight of the edge. No matter what the edge weights are, there always exists a feasible vertex labelling; one such is the function $l$ given by

$$\left.\begin{array}{ll} l(x) = \max\limits_{y \in y} w(xy) & \text{if } x \in X \\ l(y) = 0 & \text{if } y \in Y \end{array}\right\} \qquad \ldots(ii)$$

If $l$ is a feasible vertex labelling, we denote by $E_1$ the set of those edges for which equality holds in (i); that is

$$E_1 = \{xy \in E \mid l(x) + l(y) = w(xy)\}$$

The spanning subgraph of $G$ with edge set $E_l$ is referred to as the equality subgraph corresponding to the feasible vertex labelling $l$, and is denoted by $G_i$.

### 5.8.3 Covering to Switching Functions

In logic design of digital machines, we have to minimize the logical functions which are Boolean functions before its implementation, which can be done by using coverings of a graph of Boolean functions. To explain the graph of Boolean function and the procedures of its simplifications, we consider the following Boolean expression.

In Boolean expression, we have $P'$ (or $\bar{p}$) denote NOT $P$, $p.q$ (or $p \wedge q$) denote $p$ AND $q$, $p + q$ (or $p \vee q$) denote $p$ OR $q$.

$$\phi = xyz + x'yz + xy'z + xyz' + x'y'z + xy'z' + x'yz' + x'y'z'$$

Each term correspond to the vertices of a graph of Boolean function and the edges are between those terms that differ by only one variable. The edge between the variables in represented by means of common variables of its end vertices. The graph of f is shown in Fig. 5.22.

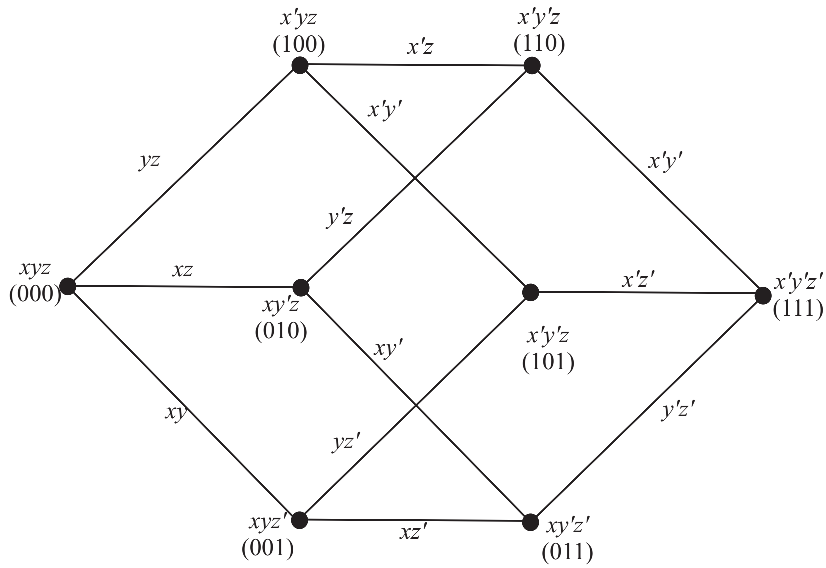



**Fig. 5.22 Covering of Switching**

One of the covering of the graph is {$yz$, $yz'$}, which is the minimal covering of the graph. Further the graph of expression for $\phi = yz + yz'$ contains only one edge between $yz$ and $yz'$. This edge is labelled by $y$ and is the minimal covering. Hence the simplified expression for $\phi$ is $\phi = y$.

### *Problem 5.4:*

*Minimize the switching function given below using Boolean graph.*

$\phi = w^1x^1y^1z^1 + w^1x^1yz^1 + wx^1y^1z^1 + w^1x^1yz + w^1xyz^1 + wxyz.$

### *Solution*:

The graph of the function f is shown below.

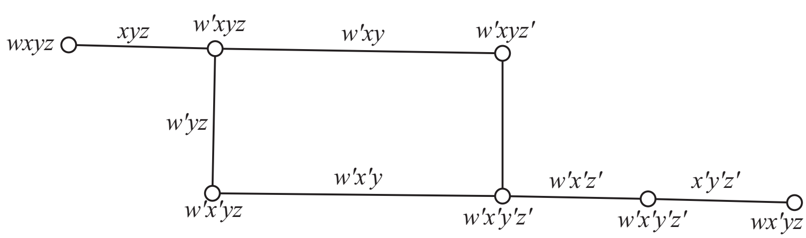



**Fig. 5.23**

One of the minimal covering of the graph is {$xyz$, $x^1y^1z^1$, $w^1xy$, $w^1x^1y$}

$\therefore$ The expression

$\phi = xyz + x^1y^1z^1 + w^1xy + w^1x^1y$ is the simplified expression of $\phi$.

To minimize further this expression $\phi$, we consider the graph of this function (As above) we get


Now the edge $w^1y$ is the minimal cover of the connected component of this graph. Considering the minimal cover with isolated vertices we get the set $\{w^1y, x^1y^1z^1, w^1x^1y\}$.

Thus the minimized expression for $\phi$ is

$$\phi = w^1y + x^1y^1z^1 + w^1x^1y.$$

We note that this expression cannot be further simplified as the graph of this function is a totally disconnected graph.

## SUMMARY

1. If every vertex in a digraph is reachable from every other vertex, the digraph is **strongly connected.**
2. Let $A = (a_{ij})$ be the adjacency matrix of a digraph $D$ with vertices $v_1, v_2, \ldots, v_n$. Let $R = A \vee A^{[2]} \vee ,\ldots, \vee A^{[n-1]} = (r_{ij})$. Then vertex $v_j$ is reachable from vertex $v_i$ if and only if $r_{ij} > 0$, where $i \neq j$. $R$. is the **reachability matrix** of the digraph.
3. A **de Bruijn sequence** for a binary alphabet is a $2^n$-bit sequence that contains every $n$-bit word as a subsequences in a cyclic fashion.
4. A vertex $v_j$ is **reachable** from a vertex $v_i$ if there is a directed path from $v_i$ to $v_j$
5. A **dag** is a digraph with no directed cycles.
6. Dags pictorially represent assignment statements and algebraic expressions with repeating subexpressions; they help find the prefix and postfix forms of such expressions.

## EXERCISES

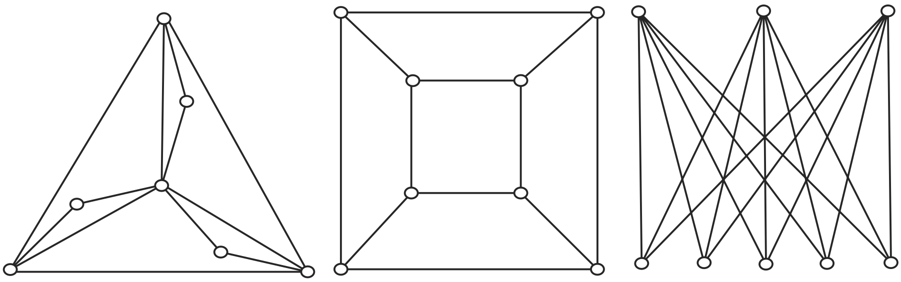
1. Show that a tree has at most one perfect matching.
2. Two people play a game on a graph G by alternately selecting distinct vertices $v_0, v_1, v_2, \ldots,$ such that, for $i >$ o, $v_i$ is adjacent to $v_{i-1.}$ The last player able to select a vertex wins. Show that the first player has a winning strategy iff $G$ has no perfect matching.
3. A $k$-factor of $G$ is a $k$-regular spanning subgraph of $G$, and $G$ is $k$-factorable if there are edge-disjoint $k$-factors $H_1, H_2, \ldots, H_n$ such that $G = H_1$ È $H_2$ È,…, È $H_{n.}$

(*a*) Show that

(*i*) $K_{n,n}$ and $K_{2n}$ are 1-factorable;

(*ii*) the Petersen graph is not 1-factorable.

(*b*) Which of the following graphs have 2-factors?



**4.** A non-negative real matrix $Q$ is doubly stochastic if the sum of the entries in each row of $Q$ is 1 and the sum of the entries in each column of $Q$ is 1. A permutation matrix is a (0, 1)-matrix which has exactly one 1 in each row and each column. (Thus every permutation matrix is doubly stochastic.) Show that

(*a*) every doubly stochastic matrix is necessarily square;

(*b*) every doubly stochastic matrix $Q$ can be expressed as a convex linear combination of permutation matrices, that is

$$Q = c_1P_1 + c_2P_2 +,\ldots,+ c_kP_k$$

where each $P_i$ is a permutation matrix, each $c_i$ is a non-negative real number, and $\sum_1^k c_i = 1$.

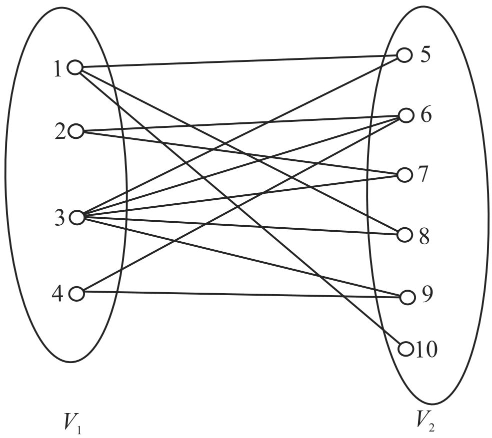
**5.** Show that a tree $G$ has a perfect matching iff $0(G - v) = 1$ for all $v \in V$.

**6.** Leg $G$ be simple, with $v$ even and $S < \frac{v}{2}$. Show that if $\in > \binom{\delta}{2} + (v - 2d - 1) + d\,(v - d)$, then $G$ has a perfect matching.

**7.** A diagonal of an $n \times n$ matrix is a set of $n$ entries no two of which belong to the same row or the same column. The weight of a diagonal is the sum of the entries in it. Find a minimum-weight diagonal in the following matrix:

$$\begin{bmatrix} 4 & 5 & 8 & 10 & 11 \\ 7 & 6 & 5 & 7 & 4 \\ 8 & 5 & 12 & 9 & 6 \\ 6 & 6 & 13 & 10 & 7 \\ 4 & 5 & 7 & 9 & 8 \end{bmatrix}$$

**8.** Show that a complete matching of $V_1$ into $V_2$ exists in the following graph.




**9.** Let $A$ be a finite set with subsets $A_1,\ldots, A_n$, and let $d_1,\ldots, d_n$ Î $N$. Show that there are disjoint subsets $D_k$ Í $A_k$, with $|D_k| = d_k$ for all $k$ £ $n$, if and only if-

$$\left|\bigcup_{i \leftarrow I} A_i\right| \geq \sum_{i \in I} d_i$$

for all $I \subseteq \{1,\ldots, n\}$.

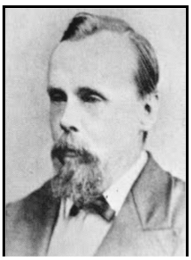
**10.** Find a cubic graph without a 1-factor.

## Suggested Readings

1. **A.V. Aho** *et al, Data Structures and Algorithms*, Addison-Wesley, Reading, MA, 1983 pp. 198-229.
2. **R.K. Ahuja** *et al.*, *Network Flows,* Prentice-Hall, Englewood Cliffs, NJ, 1993.
3. **S. Sahni**, *Concepts in Discrete Mathematics*, 2nd ed., Camelot, Fridley, MN, 1985, pp. 379-397.
4. **R.J. Wilson** and J.J. Watkins, *Graphs*: *An Introductory Approach*, Wiley, New York, 1990, pp. 80-111.
5. **S.K. Yadav**, *Elements of Graph Theory*, Ane Books, New Delhi, 2011.

❑❑❑

# 6 Colouring of Graphs



**Francis Guthrie (1831–1899)**

Francis Guthrie (1831–1899) was born in London, England. He graduated from University College London he pursued law and became a barrister. In 1850s, after colouring the countries of England on a map with four colours. He conjectured the four colour problem to his younger brother Frederick. At that time Frederick Guthrie (1833-1866) was a student of De Morgan, had showed the problem to him, who in turn communicated to William Hamilton in 1852. Hamilton did not take interest in this problem. Later on in 1878, the problem was popularized by Arthur Cayley with an announcement in a meeting of the London Mathematical society.

In 1861, Guthrie joined as a professor of mathematics at Graft-Reinet College, South Africa and later on he moved to South African University, Cape Town in 1876 where he served until death.

## 6.1 Introduction

In graph theory, Graph Colouring is a special case of graph labeling. It is an assignment of labels traditionally called “Colour’s” to elements of graph subject to certain constraint in. Its simplest form, it is a way of colouring the vertices of a graph such that no two adjacent vertices share the same colour, this is called vertex colouring. Similarly, an edge colouring assigns a colour to each edge so that no two adjacent edges share the same colour, and a face colouring of a planer graph assigns a colour to each face or region so that no two faces that share a boundary have the same colour.

Vertex colouring is the starting point of the subject and the other colouring problems can be transformed into a vertex version. For example, an edge colouring of a graph is just a vertex colouring of its line graph, and a face



S. K. Yadav, *Advanced Graph Theory*,
https://doi.org/10.1007/978-3-031-22562-8_6

colouring of a planar graph is just a vertex colouring of its planar dual. However, non-vertex colouring problems are often stand and studied as it. That is partly for perspective, and partly became some problems are best studied in non-vertex form, as for instance is edge colouring. The convention of using colours originates from colouring the countries or states of a map, where each face is titerally coloured. This was generalized to colouring the faces of a graph embedded in the plane. By planar duality it became colouring the vertices, and in this form it generalizes to all graphs. In mathematical and computer representation it is typical to use the first few positive or non-negative integers as the colours.

Generally we can use any finite set as the colour set. The nature of the colouring problem depend on the number of colours but not on what they are.

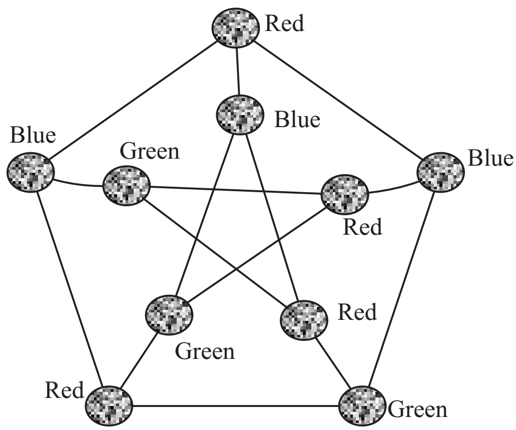



**Fig. 6.1: A Proper Vertex Colouring of the Petersen Graph with 3 Colors, the Minimum Number Possible**

Colouring of graph enjoys many practical applications as well as theoretical challenges. Beside the classical types of problems, different limitations can be set on the graph, or on the way a colour is assigned, or even on the colour itself. It has reached popularity with the general public in the form of popular number puzzle Sudoku. Graph colouring is very active in the field of research.

The first results about graph colouring deal almost exclusively with planar graphs in the form of the coloring of maps, While trying to color a map of the counties of England. **Francis Guthrie** postulated the four color conjecture, noting that four colors were sufficient to color the map so that no regions sharing a common border received the same color. Guthrie's brother passed on the question to his mathematics teacher **Augustus de Morgan** at University College, who mentioned it in a letter to **William Hamilton** in 1852. **Arthur Cayley** raised the problem at a meeting of the London Mathematical Society in 1879. The same year, **Alfred Kempe** published a paper that claimed to establish the result, and for a decade the four color problem was considered solved. For

his accomplishment Kempe was elected a Fellow of the Royal Society and later President of the London Mathematical Society.

In 1890, **Heawood** pointed out that Kempe's argument was wrong. However, in that paper he proved the five color theorem, saying that every planar map can be colored with no more than, five colors, using ideas of Kempe. In the following century, a vast amount of work and theories were developed to reduce the number of colors to four, until the four color theorem was finally proved in 1976 by **Kenneth Appel** and **Wolfgang Haken**. Perhaps surprisingly, the proof went back to the ideas of Heawood and Kempe and largely disregarded the intervening developments. The proof of the four color theorem is also noteworthy for being the first major computer-aided proof.

In 1912, **George David Birkhoff** introduced the chromatic polynomial to study the coloring problems, which was generalised to the Tutte polynomial by **Tutte**, important structures in algebraic graph theory. Kempe had already drawn attention to the general, non-planar case in 1879, and many results on generalisations of planar graph coloring to surfaces of higher order followed in the early 20th century.

In 1960, **Claude Berge** formulated another conjecture about graph coloring, the strong perfect graph conjecture, originally motivated by an information-theoretic concept called the zero-error capacity of a graph introduced by **Shannon**. The conjecture remained unresolved for 40 years, until it was established as the celebrated strong perfect graph theorem in 2002 by **Chudnovsky**, **Robertson**, **Seymour**, **Thomas**.

Graph coloring has been studied as an algorithmic problem since the early 1970s: the chromatic number problem is one of Karp's 21 NP-complete problems from 1972, and at approximately the same time various exponential-time algorithms were developed based on backtracking and on the deletioncontraction recurrence of **Zykov** (1949). One of the major applications of graph coloring, register allocation in compilers, was introduced in 1981.

Painting all the vertices of a graph with colours such that no two adjacent vertices have the same colour is called the proper colouring of a graph. A graph in which every vertex has been assigned a colour according to a proper colouring is called a properly coloured graph. (see Fig. 6.1).

## 6.2 Vertex Colouring

When used without any particular qualification, a colouring of graph is almost always a proper vertex colouring namely a labelling of the graph's vertices with colours such that no two vertices sharing the same edge have the same colour. Since a vertex with a loop could never be properly coloured, it is understood that graphs in this context are loopless. The terminology of using colours for

vertex lebels goes back to map colouring. Lebels like red and blue are only used when the number of colours is small, and normally it is understood that the labels are drawn from the integers {1, 2, 3,…}.

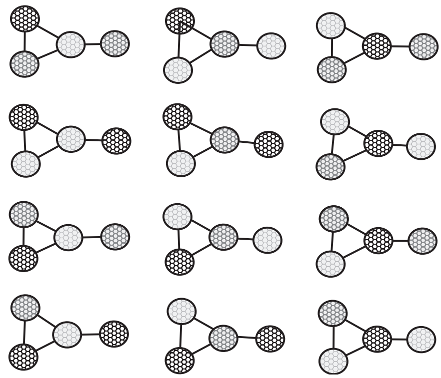


**Fig. 6.2: This Graph can be 3-coloured in 12 Different Ways**

A colouring using at most $k$ colours is called a proper $k$-colouring. The smallest number of colours needed to colour a graph $G$ is called needed to colour a graph $G$ is called its chromatic number $X(G)$. A graph that can be assigned a proper $k$-colouring is $k$-colourable, and it is $k$-chromatic of its chromatic number is exactly $k$. A subset of vertices assigned to the same colour is called a colour class, every such class forms an independent set. Thus, a $k$-colouring is the same as a partition of the vertex set into k independent sets, and the terms $k$-partite and $k$-colourable have the same meaning.

## 6.3 Chromatic Polynomial

The chromatic polynomial counts the number of ways a graph can be coloured using no more than a given number of colours. For example, using three colours, the graph in the image to the right can be coloured into 12 ways. With only two colours, it can not be coloured at all. With four colours, it can be coloured in 24+4*12 = 72 ways. There are 4! = 24 valid colourings (every assignment of four colours to any 4-vertex graph is a proper colouring) and for every choice of three of the four colours, there are 12 valid 3-colourings. So, for the graph in the example, a table of number of valid colouring would start like this:

| Available Colour | 1 | 2 | 3 | 4 | … |
|---|---|---|---|---|---|
| No of Colouring | 0 | 0 | 12 | 72 | … |

The chromatic Polynomial is a function $P(G, t)$ that counts the number of $t$-colourings of $G$. As the name indicates, for a given $G$ the function is a polynomial in it. For example the graph, $P(G, t) = t(t - 1)^2(t - 2)$ and indeed $P(G, 4) = 72$.

The Chromatic Polynomial includes at least as much information about the colourability of $G$ as does the chromatic number. Indeed, $X$ is the smallest positive integer that is not a root of chromatic polynomial.

$$X(G) = \min\{k : P(G, k) > 0\}.$$

Chromatic polynomials for certain graphs are:

| | |
|---|---|
| Triangle $K_3$ | $t(t-1)(t-2)$ |
| Complete graph $K_n$ | $t(t-1)(t-2) \ldots (t-(n-1))$ |
| Tree with $n$ vertices | $t(t-1)^{n-1}$ |
| Cycle $C_n$ | $(t-1)^n + (-1)^n(t-1)$ |
| Petersen graph | $t(t-1)(t-2)(t^7 - 12t^6 + 67t^5 - 230t^4 + 529t^3 - 814t^2 + 775t - 352)$ |

### 6.3.1 Bounds of the Chromatic Number

1. Assigning distinct colours to distinct vertices always yields a proper colouring. So,

   $$1 \leq \chi(G) \leq n.$$

2. The only graph that can be 1-coloured are edgeless graphs, and the complete graph $K_n$ of n vertices requires $\chi(K_n) = n$ colours. In the optimal colouring there must be atleast one of the graph's m edges between every pair of colour classes. So,

   $$\chi(G)\,[\chi(G) - 1] \leq 2m$$

3. If $G$ contains a clique of size $k$, then atleast $k$ colours are needed to colour that clique; in other words, the chromatic number is at least the clique number:

   $$\chi(G) \geq w(G).$$

   for interval of graphs this bound is tight.

4. The two-colourable graphs are exactly the bipertite graphs, including trees and forests. By the four colour theorem, every planar graph can be 4-colurable.
5. A greedy colouring shows that every graph can be coloured with one more colour than the maximum vertex degree, $X(G) \leq \Delta(G) + 1$.
6. Complete graphs have $\chi(G) = n$ and $\Delta(G) = n - 1$, and odd cycles have $\chi(G) = 3$ and $\Delta(G) = 2$ so, for these graphs this bound is best possible. In all other cases, the bound can be slightly improved.

### 6.3.2 Clique

A graph G is perfect iff $\chi(H) = w\ (H)$ for every induced subgraph H of G.

A clique is a set of pairarise adjacent vertices .

As usual, maximum means maximum-sized.

So, w $(\overline{H}) = \alpha(H)$

Properly coloring H means expressing V(H) as a union of cliques as H; such a set of clique in H is a clique covering of H.

Hence for every graph G we must have four optimization parameters of interest viz:

(*i*) Independence number $\alpha(G)$ [max. size of a stable set]

(*ii*) Clique number $\omega(G)$ [max. size of a clique]

(*iii*) Chromatic number $\chi(G)$ [min. size of a coloring]

(*iv*) Clique covering number $\theta(G)$ [min. size of a clique covering]

A clique of a graph is a set of mutually adjacent vertices, and the maximum size of a clique of a graph G. A set of vertices is a clique of a simple graph G iff it is a stable set of the complement $\overline{G}$

*i.e.*. $\omega(G) = \alpha(\overline{G})$

Hence any assertion about stable sets can be restated in terms of cliques or coverings.

■ **Example 6.1**: *Find the chromatic polynomial of the graphs given in Fig. 6.3*.

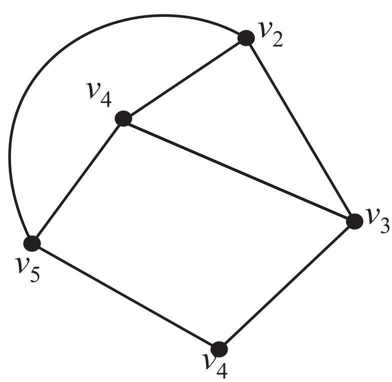



**Fig. 6.3: A 3-Cromatic Graph**

***Solution***:

We have

$$P_5(\lambda) = C_1\lambda + C_2\frac{\lambda(\lambda-1)}{2!} + C_3\frac{\lambda(\lambda-1)(\lambda-2)}{3!}$$

$$+C_4\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!} + C_5\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{5!}$$

The graph has a triangle, so, it will require at least three different colours for proper colouring.

Therefore, $C_1 = C_2 = 0$ and $C_5 = 5!$

to evaluate $C_3$, we consider three colours $x$, $y$ and $z$ These three colours can be assigned properly to vertices $v_1$, $v_2$ and $v_3$ in 3!(= 6) different ways.

After doing that, no more choice is left because $v_5$ must have the same colour as $v_3$ and $v_4$ must have as $v_2$.

Therefore $\qquad C_3 = 6$

Similarly, with four colours $v_1$, $v_2$ and $v_3$ can be properly coloured in 4*6 (= 24) different ways.

The fourth colour can be assigned to $v_4$ and $v_5$ with two choices. The fifth vertex provides us additional choice.

Therefore, $C_4 = 24*2 = 48$

making substitution in $P_5(\lambda)$, we get

$P_5(\lambda) = \lambda(\lambda - 1)(\lambda - 2) + 2\lambda(\lambda - 1)(\lambda - 2)(\lambda - 3) + \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)(\lambda - 4)$

$= \lambda(\lambda - 1)(\lambda - 2)(\lambda^2 - 5\lambda + 7)$

The presences of factors $\lambda - 1$ and $\lambda - 2$ indicates that $G$ has at least 3-chromatic. ■

■ **Example 6.2:** *Find the chromatic number for the graph $K_{3,3}$.*

***Solution***:

The chromatic polynomial for $K_{3,3}$ is given by $\lambda(\lambda-1)^5$

Thus the chromatic number of this graph is 2. (see Fig. 6.4)

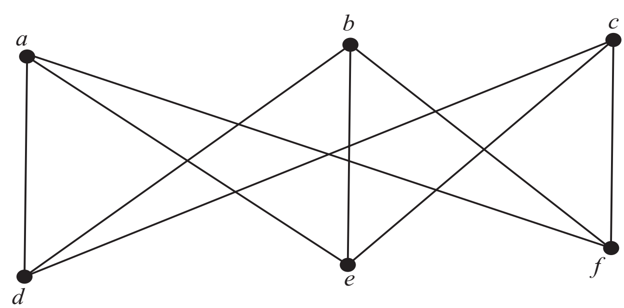



**Fig 6.4**

The vertices $a$, $b$ and $c$ may have the same colour, as they are not adjacent.

Similarly, vertices $d$, $e$ and $f$ can be coloured in proper way using one colour.

But a vertex from $\{a, b, c\}$ and a vertex from $\{d, e, f\}$ both can not have the same colour.

*i.e. every chromatic number of any bipertite graph is always 2.* ■

### *Theorem 6.1:*

*Let $\Delta(G)$ be the maximum of degrees of the vertices of a graph G. Then $\chi(G) \leq 1 + \Delta(G)$.*

### *Proof*:

The theorem can be proved by induction on $V$ *i.e.* the vertices of the graph.

When $V = 1$, $\Delta(G) = 0$ and $\chi(G) = 1$, (holds)

Let $K$ be an $\forall$ *integer* $K \geq 1$, and assume that the result holds for all graphs with $V = K$ vertices. Suppose $G$ is a graph with $(K + 1)$ vertices

Let $v$ be any vertex of $G$ and let $G_0 = \frac{G}{\{v\}}$ be the subgraph with $v$ deleted.

We note that $\Delta(G_0) \leq \Delta(G)$

Then $G_0$ has $K$ vertices. We can use the induction hypothesis to conclude that $X(G_0) \leq 1 + \Delta(G_0)$

Then, $\chi(G_0) \leq 1 + \Delta(G_0)$, so $G_0$ can be coloured with atmost $[1 + \Delta(G)]$ colours.

Since there are atmost $\Delta(G)$ vertices adjacent to $v$, one of the variable $[1 + \Delta(G)]$ colours remains for $v$. Thus, $G$ can be coloured with atmost $[1 + \Delta(G)]$ colours. ■

### *Theorem 6.2:*

*The minimum number of hours for a schedule of committee meetings in one scheduling problem is $\chi(G_0)$.*

### *Proof*:

We consider $\chi(G_0) = K$.

and the colours used in colouring $G_0$ are 1, 2, 3,…, $K$. firstly we assest that all committees can be scheduled in $K$ one-hour time periods.

We also consider all these vertices coloured 1(say) and the committees corresponding to these vertices. Since no two vertices coloured 1 are adjacent, no two such committees contain the same member.

Hence, all these committees can be scheduled to meet at the same time.

Thus, all committees corresponding to same colour vertices can meet at the same time.

Therefore, all committees can be scheduled to meet during $K$ time periods.

Further, we show that all committees cannot be scheduled in less than $K$ hours. We can prove this by contradiction. Suppose that we can schedule the committees in m one hour time periods, where $K > M$

So, we can give $G_0$ as m-colouring by colouring with the same colour all vertices which correspond to committees meeting at the same time.

To see that this is a legitimate m-colouring of $G_0$.

We consider two adjacent vertices.

These vertices correspond to two committees containing one or more common members.

Hence, these committees meet at different times, and thus the vertices are coloured differently.

However, an $m$-colouring of $G_0$ gives a contradiction since, we have $\chi(G_0) = K$. ■

## 6.4 Exams Scheduling Problem

This problem aims "How can the final exams at a university be scheduled so that no student has two exams at the same time?"

This scheduling problem can be solved by using a graph model, with vertex colouring. In this problem vertices represent the course and with an edge between two vertices of there is a common student in the courses they represent. Each time slot for a final exams represented by different colour. A scheduling of exams corresponds to a colouring of the associated graph.

### *Suppose there are seven finals to be scheduled*

Suppose the colours are numbered 1 through 7 and the following pairs of colours have common students:

1 and 2, 1 and 3, 1 and 4, 1 and 7, 2 and 3, 2 and 4, 2 and 5, 2 and 7, 3 and 4, 3 and 6, 3 and 7, 4 and 5, 4 and 6, 5 and 6, 5 and 7, and 6 and 7. (see Fig. 6.5)

A scheduling consists of a colouring of this graph. Since the chromatic number of this graph is 4 so, four times slots are needed.

A colouring of graph using four colours and the associated schedule are shown in Fig. 6.6.

| **Time** | **Courses** |
|---|---|
| (I) | 1, 6 |
| (II) | 2 |
| (III) | 3, 5 |
| (IV) | 4, 7 |

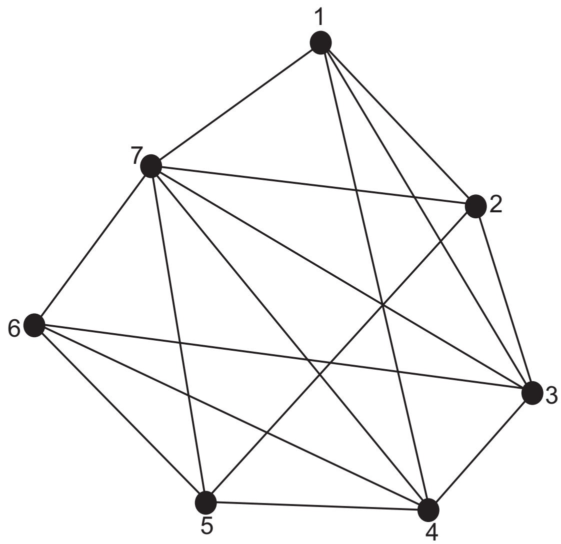



**Fig. 6.5: Scheduling Problem**

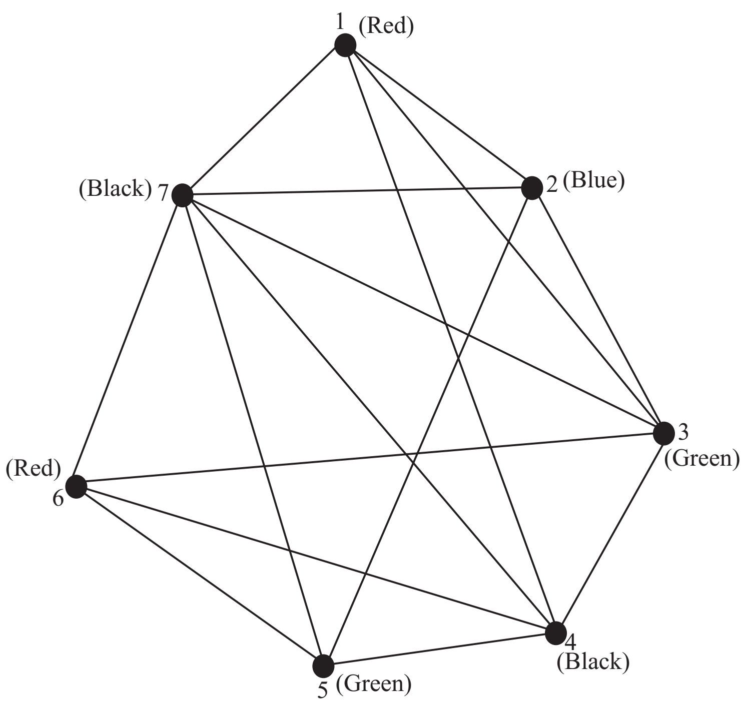



**Fig. 6.6: Colouring Schedule**

Using colouring to schedule final exams.

■ ***Example 6.3:*** *What is the chromatic number of the complete bipertite graph* $K_{m,\,n}$ *where m and n are positive integers?*

***Solution.***

In this problem, the number of colours needed may seem to depend upon $m$ and $n$.

However, only two colours are needed. Colour the set of $m$-vertices with one colour and the set of $n$-vertices with a second colour.

Since edge can not only a vertex from the set of $m$-vertices and a vertex from the set of $n$-vertices, no two adjacent vertices have the same colour.

We can take a colouring of $K_{3,4}$ with two colours as shown in Fig. 6.7.

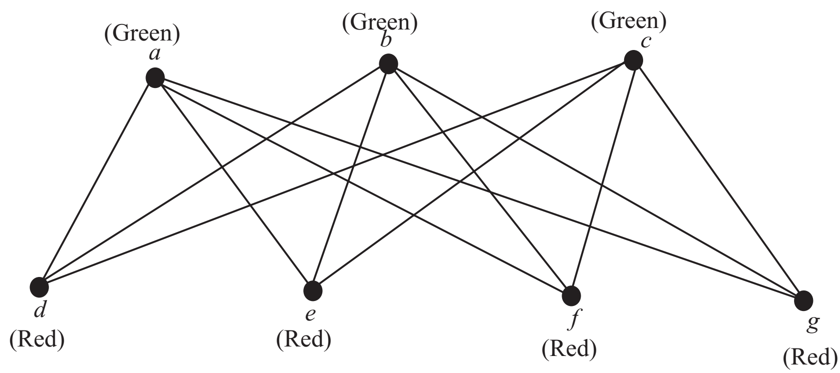



**Fig. 6.7 A colouring of $K_{3,4}$.**

A colouring of $K_n$ can be constructed using is colours by assigning a different colour to each vertex. There is no colouring of fewer colours. No two vertices can be assigned the same colour, since every two vertices of this graph are adjacent.

Hence, the chromatic number of $K_n = n$.

we can see the colouring of $K_5$ in Fig. 6.8

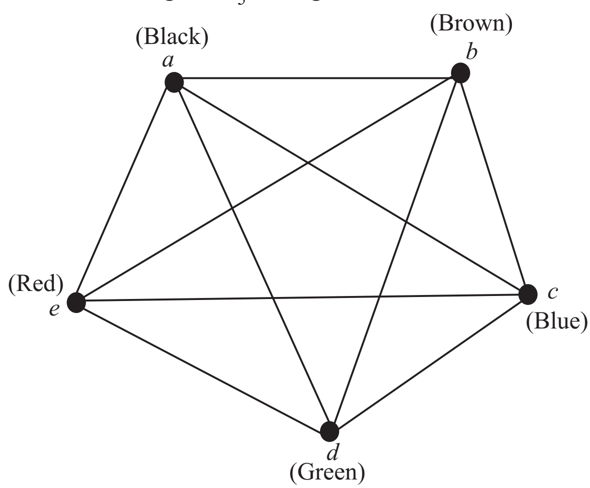



**Fig. 6.8 A colouring of $K_5$.** ■

***Theorem 6.3:***

*If G is k-critical, then $d \geq k - 1$.*

***Proof***:

This theorem can be proved by contradiction. If possible, let $G$ be a $k$-critical graph with $\delta < k - 1$, and let $v$ be a vertex of degree $\delta$ in $G$.

Since $G$ is $k$-critical, $(G - v)$ is $(k - 1)$-colourable

Let $(V_1, V_2, V_3, \ldots, V_{k-1})$ be a $(k-1)$-colouring of $(G - v)$.

By definition, $v$ is adjacent us $G$ to $\delta < (k - 1)$ vertices, and, therefore $v$ must be nonadjacent in $G$ to every vertex of some $V_j$. But then $(V_1, V_2, V_3, \ldots, V_j) \cup \{v\}, \ldots, V_{k-1})$ is a $(k-1)$-colouring of $G$. (a contradiction).

Hence $\delta \geq k - 1$. ■

***Theorem 6.4:***

*In a critical graph, no vertex cut is a clique.*

***Proof***:

This result is to be proved by contradiction. Let $G$ be a $k$-critical graph, and suppose that $G$ has a vertex cut $S$ that is a clique. We denote the $S$-component of $G$ by $G_1, G_2, G_3, \ldots, G_n$. Since $G$ is $k$-critical, each $G_i$ is $(k-1)$-colourable. Furthermore, because $S$ is a clique, the vertices in $S$ must receive distinct colours in any $(k-1)$-colouring of $G_i$. It follows that there are $(k-1)$-colourings of $G_1, G_2, G_3, \ldots, G_n$ which agree on $S$.

But these colourings together yield a $(k-1)$-colouring of $G$, which is a contradiction. ■

***Theorem 6.5:***

*If G is simple, then $\pi_k(G) = \pi_k(G - e) — \pi_k(G \cdot e)$ for any edge e of G.*

***Proof***:

Let $u$ and $v$ be the ends of $e$. To each $k$-colouring of $(G - e)$ that assigns the same colour to $u$ and $v$, there corresponds a $k$-colouring of $(G \cdot e)$ in which the vertex of $(G \cdot e)$ formed by identifying. $u$ and $v$ is assigned the common colour of $u$ and $v$. This correspondence is clearly a bijection (refer Fig. 6.9). Therefore $\pi_k(G \cdot e)$ is precisely the number of $k$-colourings of $(G - e)$ in which $u$ and $v$ are assigned the same colour.

Also, since each $k$-colouring of $(G - e)$ that assigns different colours to $u$ and $v$ is a $k$-colouring of $G$.

Conversely, $\pi_k(G)$ is the number of $k$-colourings of $(G - e)$ in which $u$ and $v$ are assigned different colours. It follows that $\pi_k(G - e) = \pi_k(G) + \pi_k(G \cdot e)$

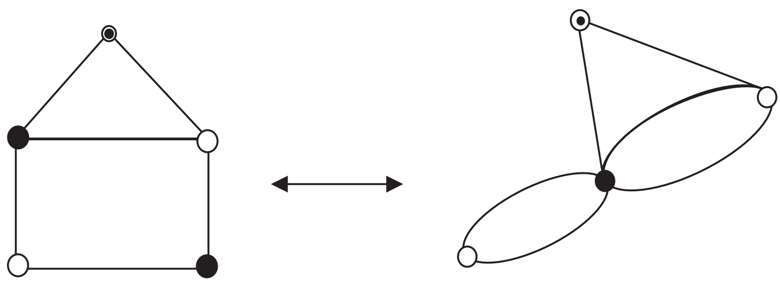


**Fig. 6.9**

■

■ ***Example 6.5:*** *Suppose that in one particular semester there are students taking each of the following combinations of courses.*

- *Mathematics, English, Biology, Chemistry*
- *Mathematics, English, Computer Science, Geography*
- *Biology, Psychology, Geography, Spanish*
- *Biology, Computer Science, History, French*
- *English, Psychology, History, Computer Science*
- *Psychology, Chemistry, Computer Science, French*
- *Psychology, Geography, History, Spanish.*

What is the minimum number of examination periods required for exams in the ten courses specified so that students taking any of the given combinations of courses have no conflicts ?

Find a possible schedule which uses this minimum number of periods.

***Solution*:**

In order to picture the situation, we draw a graph with ten vertices labeled $M$, $E$, $B$, ... corresponding to Mathematics, English, Biology and so on, and join two vertices with an edge if exams in the corresponding subjects must not be scheduled together.

The minimum number of examination periods is evidently the chromatic number of this graph. What is this ? Since the graph contains $K_5$ (with vertices $M$, $E$, $B$, $G$, $CS$), at least five different colours are needed. (The exams in he subjects which these vertices represent must be scheduled at different times). Five colours are not enough, however, since $P$ and $H$ are adjacent to each other and to each of $E$, $B$, $G$ and $CS$.

The chromatic number of the graph is, infact 6.

In Figure (6.10), we show a 6-colouring and the corresponding exam schedule.

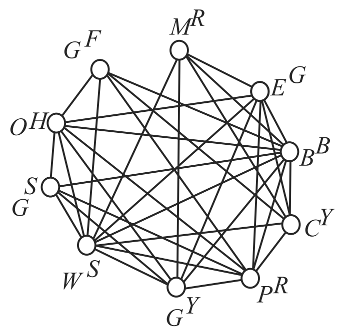



**Fig. 6.10**

| | |
|---|---|
| Period 1 | Mathematics, Psychology |
| Period 2 | English, Spanish, French |
| Period 3 | Biology |
| Period 4 | Chemistry, Geography |
| Period 5 | Computer Science |
| Period 6 | History |

### *Theorem 6.6:*

*For any graph G,* $\chi(G) \leq 1 + \max \delta(G')$,

Where the maximum is taken over all induced subgraph $G'$ of $G$.

### *Proof*:

The result is obvious for totally disconnected graphs.

Let $G$ be an arbitrary $n$-chromatic graph, $n \geq 2$.

Let $H$ be any smallest induced subgraph such that $\chi(H) = n$

The graph $H$ therefore has the property that

$\chi(H - v) = n - 1$ for all its points $v$.

It follows that deg $v \geq n - 1$ so that $\delta(H) \geq n - 1$ and hence

$n - 1 \leq \delta(H) \leq \max \delta(H') \leq \max \delta(G')$

The first maximum taken over all induced subgraphs $H'$ of $H$ and the second over all induced subgraphs $G'$ of $G$.

This implies that

$\chi(G) = n < 1 + \max \delta(G')$ ■

### *Corollary*:

For any graph $G$, the chromatic number is atmost one greater than the maximum degree $\chi \leq 1 + \Delta$.

### *Theorem 6.7:*

*For any graph G, the sum and product of* $\chi$ *and* $\chi$ *satisfy the inequalities:*

$$2\sqrt{P} \leq \chi + \overline{\chi} \leq P + 1.$$

$$P \leq \chi\overline{\chi} \leq \left(\frac{P+1}{2}\right)^2$$

### *Proof*:

Let $G$ be $n$-chromatic and let $V_1$, $V_2$,...$V_m$ be the colour classes of $G$, where $|V_i| = P_i$

Then of course $\sum P_i = P$ and max $P_i \geq \frac{P}{n}$.

Since each $V_i$ induces a complete subgraph of $\overline{G}$

$\overline{\chi} \geq$ max $P_i \geq \frac{P}{n}$ so that $\chi\overline{\chi} \geq 2\sqrt{P}$.

This establishes both lower bounds.

To show that $\chi + \overline{\chi} \leq P + 1$, we use induction on $P$, noting that equality holds when $P = 1$.

We thus assume that $\chi(G) + \chi(G) \leq P$ for all graphs $G$ having $P - 1$ points.

Let H and $\overline{H}$ be complementary graphs with P points, and let $v$ be a point of H.

Let $H$ and $\overline{H}$ be complementary graphs with $P$ points, and let $v$ be a point of $H$.

Then $G = H - v$ and $\overline{G} + \overline{H} - v$ are complementary graphs with $P - 1$ points.

Let the degree of $v$ in $H$ be $d$ so that the degree of $v$ in $\overline{H}$ is $P - d - 1$. It is obvious that

$\chi(H) \leq \chi(G) + 1$ and $\overline{\chi}(H) \leq \chi(G) + 1$

If either

$\chi(H) < \chi(G) + 1$ or $\overline{\chi}(H) < \overline{\chi}(G) + 1$.

then $\chi(H) + \overline{\chi}(H) \leq P + 1$.

Suppose then that $\chi(H) = \chi(G) + 1$ and $\overline{\chi}(H) = \overline{\chi}(G) + 1$.

This implies that the removal of $v$ from $H$, producing $G$, decreases the chromatic number so that $d \geq \chi(G)$.

Similarly $\quad P - d - 1 \geq \overline{\chi}(G)$,

thus $\quad \chi(G) + \overline{\chi}(G) \leq P - 1$

Therefore, we always have

$$\chi(H) + \overline{\chi}(H) \leq P + 1$$

Finally, applying the inequality

$$4\chi\bar{\chi} \leq (\chi + \bar{\chi})^2 \text{ we see that}$$

$$\chi\bar{\chi} \leq \left[\frac{(P+1)}{2}\right]^2$$

■

***Theorem 6.8:***

If $G$ is a graph with $n$ vertices and degree $\delta$ then $\chi(G) \geq \dfrac{n}{n-\delta}$.

***Proof*:**

Recall that $\delta$ is the minimum of the degrees of vertices.

Therefore, every vertex $v$ of $G$ has atleast $\delta$ number of vertices adjacent to it.

Hence there are at most $n - \delta$ vertices can have the same colour.

Let $K$ be the least number of colours with which $G$ can be properly coloured. Then $K = \chi(G)$.

Let $\alpha_1, \alpha_2, \ldots, \alpha_K$ be these colours and let $n_1$ be the number of vertices having colour $\alpha_1$, $n_2$ be the number of vertices having colour $\alpha_2$ and so on, and finally $n_K$ be the number of vertices having colour $\alpha_K$.

Then $n_1 + n_2 + n_3 + ,\ldots, n_k = n$ ...(1)

and $n_1 \leq n - \delta, n_2 \leq n - \delta, \ldots, n_k \leq n - \delta$ ... (2)

Adding the $K$ in equalities in (2), we obtain

$n_1 + n_2 + ,\ldots, n_k \leq K(n - d)$

or$n \leq K(n - d)$, using (1)

Since $K = \chi(G)$, this becomes

$$\chi(G) \geq \frac{n}{n-\delta}$$

■

## 6.5 Edge Colouring

A $k$-edge colouring of a loopless graph $G$ is an assignment of $k$ colours, 1, 2, 3, ..., $k$, to the edges of $G$. The colouring is proper if no two adjacent edges have the same colour.

Alternatively, a $k$-edge colouring can be thought of as a partition $(E_1, E_2, E_3, \ldots, E_k)$ of $E$ where $E_i$ denotes the possibly empty subset of $E$ assigned colour $i$. A proper $k$-edge colouring is than a $k$-edge colouring $(E_1, E_2, E_3, \ldots, E_k)$ in which each subset $E_i$ is a matching.

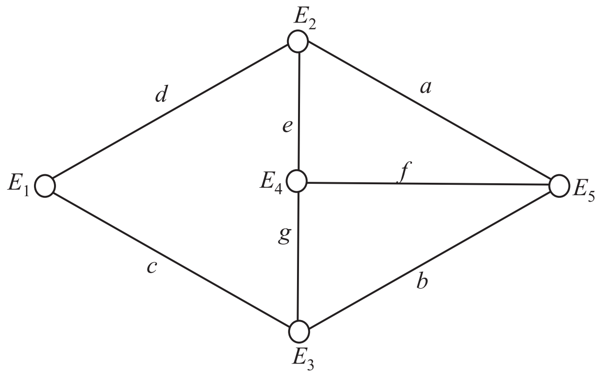



**Fig. 6.11: Edge Colouring**

The proper four edge colouring for Fig. 6.11 is ({$a$, $g$}, {$b$, $e$}, {$c$,$f$}, {$d$})

$G$ is $k$-edge-colourable if $G$ has a proper $k$-edge colouring.

Trivially, every loopless graph $G$ is $E$-edge colourable.

If $G$ is $k$-edge-colourable, then $G$ is also $l$-edge-colourable for every 1 > $k$. The edge chromatic number $\chi'(G)$, of a loopless graph $G$, is the minimum $k$ for which $G$ is $k$-edgecolourable. $G$ is $k$-edge-chromatic if $\chi'(G) = k$. It can be readily verified that the graph of Fig.6.11 has no proper 3-edge colouring. This graph is therefore 4-edge-chromatic.

Clearly, in any proper edge colouring, the edges incident with any one vertex must be assigned different colours. It follows that

$$\chi' \geq \Delta \qquad \text{...(i)}$$

We shall show that, in the case when $G$ is bipertite,

$\chi' = \Delta$. we can say that colour $i$ is represented at vertex $v$ if same edge incident with $v$ has colour $i$.

### *Lemma 6.1:*

*Let G be a connected graph that is not an odd cycle. Then G has a 2-edge colouring in which both colours are represented at each vertex of degree at least two.*

### *Proof*:

We may clearly assume that $G$ is nontrivial. Suppose, first, that $G$ is eulerian. If $G$ is an even cycle, the proper 2-edge colouring of $G$ has the required property. Otherwise, $G$ has a vertex $v_0$ of degree at least four. Let $v_o e_1 v_1, \ldots, e.v_0$ be an Euler tour of $G^*$ and set

$$E_1 = \{e_i \mid i \text{ odd}\}$$

and $$E_2 = \{e_i \mid i \text{ even}\} \qquad \text{...(i)}$$

Then the 2-edge colouring ($E_,$, $E_2$) of $G$ has the required property, since each vertex of $G$ is an internal vertex of $v_0e_1v_1$ ,..., $e_\in$,$v_0$.

If $G$ is not eulerian, construct a new graph $G^*$ by adding a new vertex $v_0$ and joining it to each vertex of odd degree in $G$. Clearly $G^*$ is eulerian. Let $v_0e_1$, $v_1$ ,..., $e_iv_0$ be an Euler tour of $G^*$ and define $E_1$ and $E_2$ as in ( is then easily verified that the 2-edge colouring ($E_1$, Ç $E$, $E_2$ Ç $E$) of $G$ has the required property. ■

### *Lemma 6.2:*

*SLet $\mathscr{C}$ = ($E_1$, $E_2$,..., $E_k$) be an optimal k-edge colouring of G. If there is a vertex u, in G and, colours i and j such that i is not represented at u and j is represented at least twice at u, then the component of $G[E_i, \cup E_j]$ that contains u is an odd cycle.*

***Proof*:**

Let $u$ be a vertex that satisfies the hypothesis of the lemma, and denote by $H$ the component of $G[E_i\ U\ E_j]$ containing $u$. Suppose that $H$ is not an odd cycle. Then, by lemma 6.1, $H$ has a 2-edge colouring in which both colours are represented at each vertex of degree at least two in $H$. When we recolour the edges of $H$ with colours $i$ and $j$ in this way, we obtain a new $k$-edge colouring $\mathscr{C}$.

$b'$= ($E'_1$, $E'_2$,..., $E'_k$) of $G$. Denoting by $c'(v)$ the number of distinct colours at $v$ in the colouring $\mathscr{C}$.

we have

$$c'(u) = c(u) + 1$$

since, now, both $i$ and $j$ are represented at $u$, and also

$$c'(v) \geq c(v) \text{ for } v \neq u$$

Thus $\sum_{v\in V} c'(v) > \sum_{v\in V} c(v)$, contradicting the choice of C.

It follows that $H$ is indeed an odd cycle ■

### *Theorem 6.9: (König's Theorem) (1916)*

*Every bipartite graph G satisfies $\chi'(G) = \Delta(G)$.*

### *Proof*:

We apply induction on $||G||$. For $||G|| = 0$ the assertion holds. Now assume that $||G|| \geq 1$, and that the assertion holds for graphs with fewer edges. Let $\Delta := \Delta(G)$, pick an edge $xy \in G$, and choose a $\Delta$edge-colouring of $(G - xy)$ by the induction hypothesis. Let us refer to the edges coloured $\alpha$ as $\alpha$-edges, etc.

In $(G - xy)$ each of $x$ and $y$ is incident with at most $(\Delta - 1)$ edges. Hence there are $\alpha, \beta \in \{ 1,..., \Delta)$ such that $x$ is not incident with an $\alpha$-edge and $y$ is not incident with a ($\beta$-edge. If $\alpha = \beta$, we can colour the edge $xy$ with this colour and are done; so we may assume that $\alpha \neq \beta$, and that $x$ is incident with a $\beta$-edge.

Let us extend this edge to a maximal walk $W$ whose edges are coloured b and a alternately. Since no such walk contains a vertex twice (why not?), $W$ exists and is a path. Moreover, $W$ does not contain $y$: if it did, it would end in $y$ on an a-edge (by the choice of b) and thus have even length, so $(W + xy)$ would be an odd cycle in $G$. We now recolour all the edges on $W$, swapping a with b. By the choice of a and the maximality of $W$, adjacent edges $(G - xy)$ are still coloured differently. We have thus found a $D$-edge-colouring of $(G - xy)$ in which neither $x$ nor $y$ is incident with a b-edge. Colouring $xy$ with b, we extend this colouring to a $D$-edge-colouring of $G$. ■

### *Theorem 6.10: (Vizing Theorem, 1964)*

*Every graph G satisfies*

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1.$$

### *Proof*:

We prove the second inequality by induction on $||G||$. For $||G|| = 0$ it is trivial. For the induction step let $G = (V, E)$ with $\Delta := \Delta(G) > 0$ be given, and assume that the assertion holds for graphs with fewer edges. Instead of '$(\Delta + 1)$-edge-colouring' let us just say '*colouring*'. An edge coloured $\alpha$ will again be called an $\alpha$-*edge*.

For every edge $e \in G$ there exists a colouring of $(G - e)$ by the induction hypothesis. In such a colouring, the edges at a given vertex $v$ use at most $d(v) \leq \Delta$ colours, so some colour $\beta \in \{1,\ldots, \Delta + 1\}$ is missing at $v$. For any other colour a, there is a unique maximal walk (possibly trivial) starting at $v$, whose edges are coloured alternately $\alpha$ and $\beta$. This walk is a path; we call it the $\alpha/\beta$ - path from $v$.

Suppose that $G$ has no colouring. Then the following holds:

*Given $xy \in E$, and any colouring of $(G - xy)$ in which the colour $\alpha$ is missing at $x$ and the colour $\beta$ is missing at $y$,(1) the $\alpha/\beta$-path from $y$ ends in $x$.*

Otherwise we could interchange the colours $\alpha$ and $\beta$ along this path and colour $xy$ with $\alpha$, obtaining a colouring of $G$ (contradiction).

Let $xy_0 \in G$ be an edge. By induction, $G_0 := G - xy_0$ has a colouring $c_0$. Let $\alpha$ be a colour missing at $x$ in this colouring. Further, let $y_0, y_1,\ldots, y_k$ be a maximal sequence of distinct neighbours of $x$ in $G$, such that $c_0(xy_i)$ is missing in $c_0$ at $y_{i-1}$ for each $i = 1,\ldots, k$. For each of the graphs $G_i := G - xy_i$ we define a colouring $c_i$, setting

$$c_i(e) := \begin{cases} c_0(xy_{j+1}) & \textit{for } e = xy_j \textit{ with } j \in \{0,\ldots,i-1\} \\ c_0(e) & \textit{otherwise}; \end{cases}$$

note that in each of these colourings the same colours are missing at $x$ as in $c_0$.

Now let β be a colour missing at $y_k$ in $c_0$. Clearly, β is still missing at $y_k$ in $c_k$. If β were also missing at $x$, we could colour $xy_k$ with β and thus extend $c_k$ to a colouring of $G$. Hence, $x$ is incident with a β-edge (in every colouring). By the maximality of $k$, therefore, there is an $i \in \{1,\ldots,k - 1\}$ such that

$$c_0(xy_i) = b.$$

Let $P$ be the α/β-path from $yk$ in $G_k$ (with respect to $c_k$; Fig. 6.12). By (1), $P$ ends in $x$, and it does so on a β-edge, since α is missing at $x$. As $\beta = c_0(xy_i) = c_k(xy_{i-1})$, this is the edge $xy_{i-1}$. In $c_0$, however, and hence also in $c_{i-1}$, β is missing at $y_{i-1}$ (by (2) and the choice of $y_i$); let $P'$ be the α/β-path from $y_{i-1}$ in $G_{i-1}$ (with respect to $c_{i-1}$). Since $P'$ is uniquely determined, it starts with $y_{i-1}Py_k$; note that the edges of $P_x$ are coloured the same in $c_{i-1}$ as in $c_k$. But in $c_0$, and hence in $c_{i-1}$, there is no β-edge at $y_k$ (by the choice of β). Therefore $P'$ ends in $y_k$, contradicting (1).

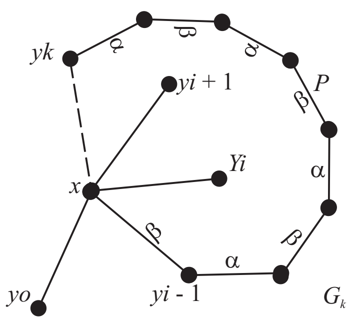



**Fig. 6.12: The α /β -path *P* in *G*$_k$**

***Note*:**

*Vizing's theorem divides the finite graphs into two classes according to their chromatic index; graphs satisfying χ′ = Δ are called (imaginatively) class 1, those with χ′ = Δ + 1 are class 2.* ■

## 6.6 List Colouring

Suppose we are given a graph $G = (V, E)$, and for each vertex of $G$ a list of colours permitted at that particular vertex: when can we colour $G$ (in the usual sense) so that each vertex receives a colour from its list? More formally, let $(S_v)_{v\in V}$ be a family of sets. We call a vertex colouring $c$ of $G$ with $c(v) \in S_v$, for all $v \in V$ a colouring from the lists $S_v$. The graph $G$ is called $k$-list-colourable, or $k$-choosable, if, for every family $(S_v)_{v\in V}$ with $|S_v| = k$ for all $v$, there is a vertex colouring of $G$ from the lists $S_v$. The least integer $k$ for which $G$ is $k$-choosable is the list-chromatic number, or choice number ch($G$) of $G$.

List-colourings of edges are defined analogously. The least integer $k$ such that $G$ has an edge colouring from any family of lists of size $k$ is the list-chromatic index ch′($G$) of $G$; formally, we just set $\mathrm{ch}'(G) := \mathrm{ch}(L(G))$, where $L(G)$ is the line graph of $G$.

In principle, showing that a given graph is $k$-choosable is more difficult than proving it to be $k$-colourable: the latter is just the special case of the former where all lists are equal to $\{1,\ldots, k\}$. Thus,

$$\mathrm{ch}(G) \geq \chi(G) \text{ and } \mathrm{ch}'(G) \geq \chi'(G)$$

for all graphs $G$.

***List colouring conjecture.*** *Every graph G satisfies ch′(G) = χ′(G).*

We shall prove the list colouring conjecture for bipartite graphs. As a tool we shall use orientations of graphs . If $D$ is a directed graph and $v \in V(D)$, we denote by $N^+(v)$ the set, and by $d^+(v)$ the number, of vertices w such that $D$ contains an edge directed from $v$ to $w$.

To see how orientations come into play in the context of colouring, let us recall the greedy algorithm. In order to apply the algorithm to a graph $G$, we first have to choose a vertex enumeration $v_1,\ldots,v_n$ of $G$. The enumeration chosen defines an orientation of $G$: just orient every edge $v_iv_j$ 'backwards', from $v_i$ to $v_j$ if $i > j$. Then, for each vertex $v_i$ to be coloured, the algorithm considers only those edges at $v_i$ that are directed away from $v_i$: if $d^+(v) < k$ for all vertices $v$, it will use at most $k$ colours. Moreover, the first colour class $U$ found by the algorithm has the following property: it is an independent set of vertices to which every other vertex sends an edge. The second colour class has the same property in $G - U$, and so on.

The following lemma generalizes this to orientations $D$ of $G$ that do not necessarily come from a vertex enumeration, but may contain some

if, for every vertex $v \in D - U$, there is an edge in $D$ directed from $v$ to a vertex in $U$. Note that kernels of non-empty directed graphs are themselves non-empty.

### *Lemma 6.3:*

Let $H$ be a graph and $(S_v)_{v \in V(H)}$ a family of lists. If $H$ has an orientation $D$ with $d^+(v) < |S_v|$ for every $v$, and such that every induced subgraph of $D$ has a kernel, then $H$ can be coloured from the lists $S_v$.

***Proof***:

We apply induction on $|H|$. For $|H| = 0$ we take the empty colouring. For the induction step, let-$|H| > 0$. Let $\alpha$ be a colour occurring in one of the lists $S_v$ and let $D$ be an orientation of $H$ as stated. The vertices $v$ with $\alpha \in S_v$ span a non-empty subgraph $D'$ in $D$; by assumption, $D'$ has a kernel $U \neq \emptyset$.

Let us colour the vertices in $U$ with $\alpha$, and remove $\alpha$ from the lists of all the other vertices of $D'$. Since each of those vertices sends an edge to $U$, the modified lists $S'_v$ for $v \in D - U$ again satisfy the condition $d^+(v) < |S'_v|$ in $D - U$. Since $D - U$ is an orientation of $H - U$, we can thus colour $H - U$ from those lists by the induction hypothesis. As none of these lists contains $\alpha$, this extends our colouring $U \to \{\alpha\}$ to the desired list colouring of $H$.

## 6.7 Greedy Colouring

The greedy algorithm considers the vertices in a specific order $V_1,\ldots,V_n$ and assigns to $V_i$ the smallest available color not used by $v_i$'s neighbours among $V_1,\ldots,V_{i-1}$, adding a fresh color if needed. The quality of the resulting coloring depends on the chosen ordering. There exists an ordering that leads to a greedy coloring with the optimal number of $\chi(G)$ colors. On the other hand, greedy colorings can be arbitrarily bad; for example, the crown graph on n vertices can be 2-colored, but has an ordering that leads to a greedy coloring with $n / 2$ colors.

If the vertices are ordered according to their degrees, the resulting greedy coloring uses at most $max_i$ min $\{d(x_i) + 1, i\}$ colors, at most one more than the graph's maximum degree. This heuristic is sometimes called the Welsh–Powell algorithm. Another heuristic establishes the ordering dynamically while the algorithm proceeds, choosing next the vertex adjacent to the largest number of different colors. Many other graph coloring heuristics are similarly based on greedy coloring for a specific static or dynamic strategy of ordering the vertices, these algorithms are sometimes called *sequential coloring algorithms.*

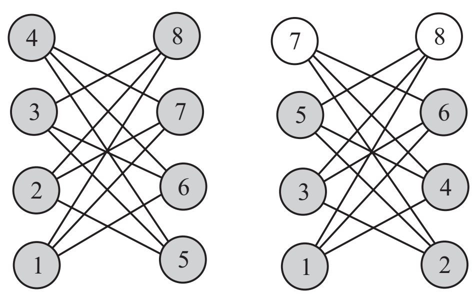



**Fig. 6.13: Two Greedy Colorings of the same Graph using Different Vertex Orders. The Right Example Generalises to 2-colourable Graphs with *n* Vertices when the Greedy Algorithm Expends *n*/2 Colours**

### *Theorem 6.11: Four Colour Problem*

*Every planar graph has a chromatic number of four or less.*

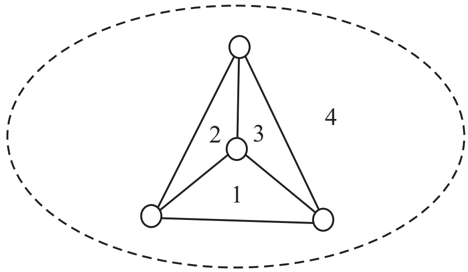



**Fig. 6.14: Four Colour Problem**

### *Proof*:

We assume the four colour conjecture holds and let $G$ be any plane map.

Let $G^*$ be the underlying graph of the geometric dual of $G$.

Since two regions of $G$ are adjacent if and only if the corresponding vertices of $G^*$ are adjacent, map $G$ is 4-colorable because graph $G^*$ is 4-colorable.

Conversely, assume that every plane map is 4-colorable and let $H$ be any planar graph.

Without loss of generality, we suppose $H$ is a connected plane graph.

Let $H^*$ be the dual of $H$, so drawn that each region of $H^*$ encloses precisely one vertex of $H$. The connected plane pseudograph $H^*$ can be converted into a plane graph $H'$ by introducing two vertices into each loop of $H^*$ and adding a new vertex into each edge in a set of multiple edges.

The 4-colorability of $H'$ now implies that $H$ is 4-colorable, completing the verification of the equivalence.

If the four color conjecture is ever proved, the result will be best possible, for it is easy to give examples of planar graphs which are 4-chromatic, such as $K_4$ and $W_6$.

### *Theorem 6.12: Five Colour Theorem*

*Every planar graph is 5-colourable.*

***Proof***:

Let $G$ be a plane graph with $n \geq 6$ vertices and $m$ edges. We assume inductively that every plane graph with fewer than $n$ vertices can be 5-coloured. We have,

$$d(G) = 2m/n \leq 2(3n - 6)/n < 6 ;$$

let $v \in G$ be a vertex of degree at most 5. By the induction hypothesis, the graph $H := G - v$ has a vertex colouring $c$: $V(H) \to (1,\ldots,5\}$. If $c$ uses at most 4 colours for the neighbours of $v$, we can extend it to a 5 colouring of $G$. Let us assume, therefore, that $v$ has exactly 5 neighbours, and that these have distinct colours.

Let $D$ be an open disc around $v$, so small that it meets only those five straight edge segments of $G$ that contain $v$. Let us enumerate these segments according to their cyclic position in $D$ as $s_1,\ldots, s_5$, and let $vv_i$ be the edge containing $s_i$ ($i = 1,\ldots, 5$; Fig. 6.15) Without loss of generality we may assume that $c(v_i) = i$ for each $i$.

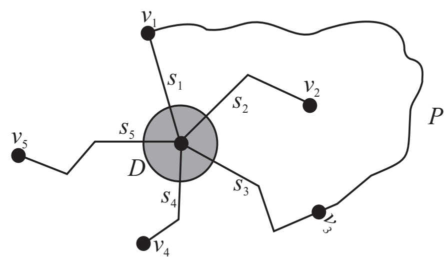



**Fig. 6.15: The Proof of the Five Colour Theorem**

Let us show first that every $v_1$–$v_3$ path $P \subseteq H$ separates $v_2$ from $v_4$ in $H$. Clearly, this is the case if and only if the cycle $C := vv_jPv_3v$ separates $v_2$ from $v_4$ in $G$. We prove this by showing that $v_2$ and $v_4$ lie in different faces of $C$.

Consider the two regions of $D \setminus (s_1 \cup s_3)$. One of these regions meets $s_2$, the other $s_4$. Since $C \cap D \subseteq s_1 \cup s_3$, the two regions are each contained within a face of $C$. Moreover, these faces are distinct: otherwise, $D$ would meet only one face of $C$, contrary to the fact that $v$ lies on the boundary of both faces. Thus $D \cap s_2$ and $D \cap s_4$ lie in distinct faces of $C$. As $C$ meets the edges $vv_2 \supseteq s_2$ and $vv_4 \supseteq s_4$ only in $v$, the same holds for $v_2$ and $v_4$.

Given $i, j \in \{1,\ldots, 5\}$, let $H_{i,j}$ be the subgraph of $H$ induced by the vertices coloured $i$ or $j$. We may assume that the component $C_1$ of $H_{1,3}$ containing $v_1$ also contains $v_3$. Indeed, if we interchange the colours 1 and 3 at all the vertices of $C_1$, we obtain another 5-colouring of $H$; if $v_3 \notin C_1$, then $v_1$ and $v_3$ are both coloured 3 in this new colouring, and we may assign colour 1 to $v$. Thus, $H_{1,3}$ contains a $v_1 - v_3$ path $P$. As shown above, $P$ separates $v_2$ from $v_4$ in $H$. Since $P \cap H_{2,4} = 0$, this means that $v_2$ and $v_4$ lie in different components of $H_{2,4}$. In the component containing $v_2$, we now interchange the colours 2 and 4, thus recolouring $v_2$ with colour 4. Now $v$ no longer has a neighbour coloured 2, and we may give it this colour.

■ **Example 6.6:** *Find all possible maximal independent sets of the following graph using Boolean expression.*

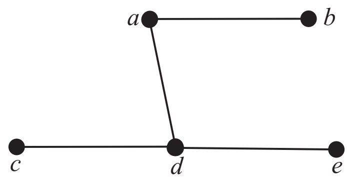



**Fig. 6.16**

***Solution***:

The Boolean expression for this graph

$$\begin{aligned}
\phi &= \Sigma xy = ab + ad + cd + de \text{ and} \\
\phi' &= (a' + b')(a' + b')(d' + e') \\
&= \{a'(a' + d') + b'(a' + d')\} \\
&\quad \{c'(d' + e') + (d'(d' + e') \\
&= \{a' + b'a' + b'd'\} \ \{c'd' + c'e' + d'\} \\
&= \{a'(1 + b') + b'd'\}\{d'(c' + 1) + (c'e'\} \\
&= \{a' + b'd'\}\{d' + c'e'\} \\
&= a'd' + a'c'e' + b'd' + b'c'd'e' \\
&= a'd' + a'c'e' + b'd' \ (1 + c'e') \\
&= a'd' + a'c'e' + b'd'
\end{aligned}$$

Thus, $\phi_1 = a'd', f_2 = a'c'e'$

and $\phi_3 = b'd'$.

Hence maximal independent sets are

$$V - \{a, b\} = \{b, c, e\}$$

$$V - \{a, c, e\} + \{b, d\}$$

and $V - \{b, d\} = \{a, c, e\}$. ■

■ **Example 6.7:** *Prove that for a graph G with n vertices*

$$\beta(G) \geq \frac{n}{\chi(G)}.$$

***Solution***:

Let $K$ be the minimum number of colours with which $G$ can be properly colored.

Then $K = \chi(G)$. Let $a_1, a_2, \ldots, a_K$ be these colours and let $n_1, n_2, \ldots, n_K$ be the number of vertices having colours $\alpha_1, \alpha_2, \ldots, \alpha_K$ respectively.

Then $n_1, n_2,\ldots, n_K$ are the orders of the maximal independent sets, because a set of all vertices having the same colour contain all vertices which are mutually non-adjacent.

Since $\beta(G)$ is the order of a maximal independent set with largest number of vertices, none of $n_1, n_2,\ldots,n_K$ can exceed $\beta(G)$.

*i.e.*, $n_1 \le \beta(G), n_2 \le \beta(G),\ldots,n_K \le \beta(G)$ ■

## 6.8 Applications

### 6.8.1 The Time Table Problem

In a school, there are m teachers $X_1, X_2,\ldots, X_m$, and $n$ classes $Y_1, Y_2,\ldots, Y_n$. Given that teacher $X_i$ is required to teach class $Y_i$ for $p_{ij}$ periods, schedule a complete timetable in the minimum possible number of periods.

The above problem is known as the timetabling problem, and can be solved completely using the theory of edge colourings developed in this chapter. We represent the teaching requirements by a bipartite graph $G$ with bipartition $(X, Y)$, where $X = \{x_1, x_2,\ldots, x_m\}$, $Y = \{y_1, y_2,\ldots, y_n\}$ and vertices $x_{\mathrm{i}}$ and $y_i$ are joined by $p_{ij}$ edges. Now, in any one period, each teacher can teach at most one class, and each class can be taught by at most one teacher—this, at least, is our assumption. Thus a teaching schedule for one period corresponds to a matching in the graph and, conversely, each matching corresponds to a possible assignment of teachers to classes for one period. Our problem, therefore, is, to partition the edges of $G$ into as few matchings as possible or, equivalently, to properly colour the edges of $G$ with as few colours as possible. Since $G$ is bipartite. Hence, if no teacher teaches for more than $p$ periods, and if no class is taught for more than $p$ periods, the teaching requirements can be scheduled in a $p$-period timetable. Furthermore, there is a good algorithm for constructing such a timetable. We thus have a complete solution to the timetabling problem.

However, the situation might not be so straightforward. Let us assume that only a limited number of classrooms are available. With this additional constraint, how many periods are now needed to schedule a complete timetable?

Suppose that altogether there are *1* lessons to be given, and that they have been scheduled in a $p$-period timetable. Since this timetable requires an average of $l/p$ lessons to be given per period, it is clear that at least $\{l/p\}$ rooms will be needed in some one period. It turns out that one can always arrange $l$ lessons in a $p$-period timetable so that at most $\{l/p\}$ rooms are occupied in any one period.

### 6.8.2 *Scheduling of Jobs*

Vertex coloring models to a number of scheduling problems. In the cleanest form, a given set of jobs need to be assigned to time slots, each job requires one such slot. Jobs can be scheduled in any order, but pairs of jobs may be in

conflict in the sense that they may not be assigned to the same tine slot. For example because they both rely on a shared resource. The corresponding graph contains a vertex for every job and an edge for every conflicting pair of jobs. The chromatic number of the graph is exactly the minimum makes an, the optimal time to finish all jobs without conflicts.

Details of the scheduling problem define the structure of the graph. For example, when assigning aircrafts to flights, the resulting conflict graph is an interval graph, so the coloring problem can be solved efficiently. In bandwidth allocation to radio stations, the resulting conflict graph is a unit disk graph, so the coloring problem is 3-approximable.

### 6.8.3 Ramsey Theory

An important class of improper coloring problems is studied in Ramsey theory, where the graph's edges are assigned to colors, and there is no restriction on the colors of incident edges. A simple example is the friendship theorem says that in any coloring of the edges of $K_6$ the complete graph of six vertices there will be a monochromatic triangle; often illustrated by saying that any group of six people either has three mutual strangers or three mutual acquaintances. Ramsey theory is concerned with generalisations of this idea to seek regularity amid disorder, finding general conditions for the existence of monochromatic subgraphs with given structure.

### 6.8.4 *Storage Problem*

A company manufactures $n$ chemicals $C_1, C_2, \ldots, C_n$. Certain pairs of these chemicals are incompatible and would cause explosions if brought into contact with each other. As a precautionary measure the company wishes to partition its warehouse into compartments, and store incompatible chemicals in different compartments. What is the least number of compartments into which the warehouse should be partitioned?

We obtain a graph $G$ on the vertex set $\{v_1, v_2, \ldots, v_n\}$ by joining two vertices $v_i$ and $v_j$ if and only if the chemicals $C_i$ and $C_j$ are incompatible. It is easy to see that the least number of compartments into which the warehouse should be partitioned is equal to the chromatic number of $G$.

The solution of many problems of practical interest (of which the storage problem is one instance) involves finding the chromatic number of a graph. Unfortunately, no good algorithm is known for determining the chromatic number. Here we describe a systematic procedure which is basically 'enumerative' in nature. It is not very efficient for large graphs.

Since the chromatic number of a graph is the least'number of independent sets into which its vertex set can be partitioned, we begin by describing a method for listing all the independent sets in a graph. Because every independent set is a subset of a maximal independent set, it suffices to determine all the maximal independent sets. In fact, our procedure first determines complements of maximal independent sets, that is, minimal coverings.

## SUMMARY

1. A graph in which every vertex has been assigned a colour according to a proper colouring is called a properly coloured graph.
2. A colouring using at most $k$ colour is called a proper $k$-colouring.
3. A Chromato Polynomial includes at least as much information about the colourability of $G$ as does the Chromatic number.
4. Every Chromatic number of any bipartite graph is always 2.
5. A minimum number of hours for a scheduled of committee meeting in one scheduling problem is $X(G_0)$.
6. If $G$ in $k$-critically, then $d \geq k$-1.
7. In a critical graph, no vertex cut is a clique.
8. A $k$-edge colouring of a loopless graph $G$ is an assignment of $k$ colours 1, 2, 3, ..., $k$, to the edge of $G$.
9. The colouring is proper if no two adjacent edges have the same colour.
10. Every planar graph is 5-colourable.

## EXERCISES

1. Describe a good algorithm for finding a proper $(\Delta + 1)$-edge colouring of a simple graph $G$.
2. Show that if $G$ is simple with d > 1, then $G$ has a $(d - 1)$-edge colouring such that all (d – 1) colours are represented at each vertex.
3. If $G$ is bipertite, then prove that c′ = D.
4. If $G$ is simple, Hence prove that either $c' = D$ or $c' = \Delta + 1$.
5. The product of simple graphs $G$ and $H$ is the simple graph $G \times H$ with vertex set $V(G) \times V(H)$, in which $(u, v)$ is adjacent to $(u', v')$ if and only if either $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$.
   (*a*) Using Vizing's theorem, show that $c'(G \times K_2) = D(G \times K_2)$.
   (*b*) Deduce that if $H$ is nontrivial with $c'(H) = D(H)$, then $c'(G \times H) = D(G \times H)$.

6. In a school there are seven teachers and twelve classes. The teaching requirements for a five-day week are given by the matrix

| | $Y_1$ | $Y_2$ | $Y_3$ | $Y_4$ | $Y_5$ | $Y_6$ | $Y_7$ | $Y_8$ | $Y_9$ | $Y_{10}$ | $Y_{11}$ | $Y_{12}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $X_1$ | 3 | 2 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| $X_2$ | 1 | 3 | 6 | 0 | 4 | 2 | 5 | 1 | 3 | 3 | 0 | 4 |
| $X_3$ | 5 | 0 | 5 | 5 | 0 | 0 | 5 | 0 | 5 | 0 | 5 | 5 |
| $X_4$ | 2 | 4 | 2 | 4 | 2 | 4 | 2 | 4 | 2 | 4 | 2 | 3 |
| $X_5$ | 3 | 5 | 2 | 2 | 0 | 3 | 1 | 4 | 4 | 3 | 2 | 5 |
| X6 | 0 | 3 | 4 | 3 | 4 | 3 | 4 | 3 | 4 | 3 | 3 | 0 |

where $p_{ij}$ is the number of periods that teacher $X_i$ must teach class $Y_j$.

(*a*) Into how many periods must a day be divided so that requirements can be satisfied?

(*b*) If an eight-period/day timetable is drawn up, how many class-rooms will be needed?

## Suggested Readings


1. **R.P. Grimaldi**, *Discrete and Combinatorial Mathematics*: *An Applied Introduction,* 4th ed., Addison-Wesley, Reading, MA, 1999 pp. 547-590.
2. **E. Horowitz** and **S. Sahni**, *Fundamentals of Computer Algorithms,* Computer Science Press, Rockville, MD, 1978, pp. 152-370.
3. **A.Tucker**, *Applied Combinatorics*, 2nd ed., Wiley, New York, NY, 1984, pp. 80-122.
4. **R.J. Wilson** and **J.J. Watkins**, *Graphs*: *An Introductory Approach*, Wiley, New York, 1990, pp. 185-214.


❑❑❑

# 7 Ramsey Theory for Graphs

**Frank P. Ramsey (1903–1930)**

Frank P. Ramsey (1903) was born in Chambridge. His father Authur S. Ramsey was also a mathematician and president of Magdalene College where Frank studied and later on shifted to Trinity College for graduation in mathematics. He studied Analytic philosophy, philosophy of mathematics, logic, metaphysics, epistemology and redundancy theory of truth as main interests. His noble ideas are Ramsey sentences, Ramsey – Lewis mathod and Ramsey theory. He died in early age of 26 on January 19, 1930.

## 7.1 Introduction

**Ramsey theory** refers to the **study of partitions of large structures**. Typical results state that **a special substructure must be occuring in some class of the partition**. **Motzkin** described that "**complete disorder is impossible**". The objects we consider are merely sets and numbers, and the techniques are little more than induction. Ramsey's theorem generalizes the pigeonthole principle, which itself concerns the partition of sets. We study the applications of the pigeonhole principle, prove Ramsey's theorem, and then focus on Ramsey-type questions for graphs. Ramsey's Theorem guarantees a special substructure. We can warmly discuss *Revised Pigeonhole Principle*. The pigeonhole principle states that if $m$ objects are partitioned into $n$ classes, then some class has at least $[m/n]$ objects. This is a discrete version of the statement that every set of a number contains a number at least as large as the average.

Among six persons it is possible to find three mutual acquaintances or three mutual non-acquaintances. In the language of graph theory, we are asked to show that for every simple graph $G$ with six vertices, there is a triangle in $G$ or



S. K. Yadav, *Advanced Graph Theory*,
https://doi.org/10.1007/978-3-031-22562-8_7

in $\overline{G}$ . The degree of vertex $x$ in $G$ and $\overline{G}$ sum to 5, so the pigeonhole principle implies that one of them is at least 3.

By symmetry we may assume that $d_G(x) \geq 3$. If two neighbours of $x$ are adjacent, then they form a triangle in $G$ with $x$; otherwise, three neighbours of $x$ form a triangle in $\overline{G}$ (as shown in Fig. 7.1).

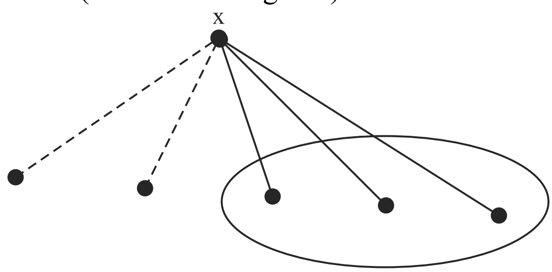



**Fig. 7.1: Ramsey Theory**

The pigeonhole principle guarantees a class with objects where we partition objects into classes. The famous theorem of **Ramsey (1930)** makes a similar statement about partitioning the $r$-element subsets of objects into classes. Ramsey's theorem says that whenever we partition the $r$-sets in a sufficiently larger set $S$ into $k$-classes, there is a $p$-subset of $S$ whose $r$-sets all line in the same class. A partition is a separation of a set into subsets, and the set we want to partition consists of subsets of another set, so far the language of partitioning. Recall that a $k$-colouring of a set is a partition of it into $k$ classes.

A class or its label is a colour. Typically we can use $[k]$ as the set of colours, in which are of $k$-colouring of $X$ can be viewed as a function $f: X \rightarrow [k]$.

Let $\binom{S}{r}$ denote the set of $r$-element subsets ($r$-sets) of a set $S$. A set $T \subseteq S$ is homogeneous under a colouring of $\binom{S}{r}$ of all $r$-sets in $T$ receive the same colour; it is $i$-homogeneous of that colour is $i$.

Let $r$ and $p_1, p_2, \ldots, p_k$ be the positive integers. If there is an integer $N$ such that every $k$-colouring of $\binom{|N|}{r}$ yields an $i$-homogeneous set of size $p_i$ for some $i$, then the smallest such integer is the Ramsey number $R(p_1, p_2, \ldots, p_k : r)$.

Ramsey's theorem states that such integer exists for every choice of $r$ and $p_1, p_2, p_3, \ldots, p_k$. When the mates all equal to $p$, the theorem states that every $k$-colouring of the $r$-sets of a sufficiently large set has a $p$-set whose $r$-sets receive the same colour.

Ramsey theorem defines the Ramsey number $R(p_1, p_2, \ldots, p_k : r) = N$, we must exhibit a $k$-colouring of the $r$-sets among N-1 points that meets no quote and we must show that every colouring on $N$ points meets the same quota. In principle, we could use a computer to examine all $k$-colourings of $\binom{|n|}{r}$ for successive $n$ until we find the first $N$ such that every such colouring meets a

quota $p_i$ for some $i$. Even for two colour Ramsey numbers, $2^{\binom{n}{2}}$ rapidly becomes too large to contemplate.

Ramsey Theorems for $r = 2$ says that k-colouring the edges of a large enough complete graph forces a monochromatic complete subgraph. A non-chromatic (subgraph) $p$-clique contains a monochromatic copy of every $p$-vertex graph. Perhaps monochromatic copies of graphs with fewer edges can be forced by colouring a smaller graph needed to force $k_p$. For example, 2-colouring the edges of $k_3$ always yields a monochromatic $P_3$, although six points are needed to force a monochromatic triangle. This suggests many Ramsey number questions, some easier to answer than the questions for cliques.

## 7.2 Independent Sets and Cliques

A subset $S$ of $V$ is called an independent set of $G$ if no two vertices of $S$ are adjacent in $G$. An independent set is maximum of $G$ has no independent set $S'$ with $|S'| > |S|$.

We recall that a subset $K$ of $V$ such that every edge of $G$ has at least one and in $K$ is called a covering of $G$.

### *Theorem 7.1:*

*A set $S \subseteq V$ is an independent set of G if and only if $V \backslash S$ is a covering of G.*

### *Proof*:

By definition, $S$ is an independent set of $G$ iff. no edge of $G$ has both ends in $S$ or, equivalently, if and only if each edge has atleast one end in $V \setminus S$. But this is so iff $V \setminus S$ is a covering of $G$.

### *Note*:

(*i*) *The number of vertices in a maximum independent set of G is called the independence number of G and is denoted by a(G); similarly, the number of vertices in a minimum covering of G is the covering number of G and is denoted by* $\beta$*(G).*

(*ii*) *The edge analogue of an independent set is a set of links no two of which are adjacent, that is, a matching. The edge analogue of a covering is called an edge covering. An edge covering of G is a subset L of E such that each vertex of G is an end of some edge in L. Note that edge coverings do not always exist; a graph G has an edge covering if and only if d > 0. We denote the number of edges in a maximum matching of G by a′(G), and the number of edges in a minimum edge covering of G by b′(G); the numbers a′(G) and b′(G) are the edge independence number and edge covering number of G, respectively.*

(*iii*) *Matchings and edge coverings are not related to one another as simply as are independent sets and coverings; the complement of a matching need not be an edge covering, nor is the complement of an edge covering necessarily a matching. However, it so happens that the parameters* $\alpha'$ *and* $\beta'$ *are related in precisely the same manner as are* $\alpha$ *and* $\beta$.

***Corollary 7.1:*** $\alpha + \beta = v$.

***Proof*:**

Let $S$ be a maximum independent set of $G$ and let $K$ be a minimum covering of $G$. Then, we have $V \setminus K$ is an independent set and $V \setminus S$ is a covering.

Therefore $\quad v - b = |V \setminus K| \leq \alpha \quad$ ...(*i*)

and $\quad v - \alpha = |V \setminus S| \geq \beta \quad$ ...(*ii*)

from (*i*) & (*ii*) we get $\alpha + \beta = v$.

## 7.3 Original Ramsey's Theorems

Ramsey's theorem says that, given an integer $r \geq 0$, every large enough graph $G$ contains either $K^r$ or $\overline{K}^r$ as an induced subgraph. This may seem surprising: we need about $(r-2)/(r-1)$ of all possible edges to force a $K^r$ subgraph in $G$, but neither $G$ nor $\overline{G}$ can be expected to have more than half the total number of edges. As the **Turan graphs** squeezing many edges into $G$ without creating a $K^r$ imposes additional structure on $G$, which may help us find an induced $\overline{K}^r$.

We try to build a $K^r$ or $\overline{K}^r$ in $G$ inductively, starting with an arbitrary vertex $v_1 \in V_1 := V(G)$. If $|G|$ is large, there will be a large set $V_2 \subseteq V_1 \setminus \{v_1\}$ of vertices that are either all adjacent to $v_1$ or all non-adjacent to $v_1$.

We may think of $v_1$ as the first vertex of a $K^r$ or $\overline{K}^r$ whose other vertices all lie in $V_2$. Let us then choose another vertex $v_2 \in V_2$ for our $K^r$ or $\overline{K}^r$. Since $V_2$ is large, it will have a subset $V_3$, still fairly large, of vertices that are all 'of the same type' with respect to $v_2$ as well: either all adjacent or all non-adjacent to it. We then continue our search for vertices inside $V_3$, and so on (Fig. 7.2).

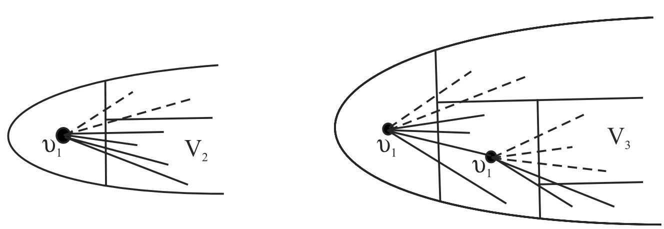



**Fig. 7.2: Choosing the Sequence $v_1$, $v_2$,..., $v_n$**

This depends on the size of our initial set $V_1$: each set $V_i$ has at least half the size of the predecessor $V_{i-1}$, so we shall be able. to complete $s$ construction steps

if $G$ has order about $2^s$. As the proof shows, the choice of $s = 2r - 3$ vertices $v_i$ suffices in order to find among them the vertices of a $K^r$ or $\overline{K}^r$.

The least integer $n$ associated with $r$ is the Ramsey number $R(r)$ of $r$; proof shows that $R(r) \leq, 2^{2r-3}$. We can use a simple probabilistic argument to show that $R(r)$ is bounded below by $2^{r/2.}$

A colouring of (the elements of) a set $X$ with $c$ colours, or $c$-colouring for short, is simply a partition of $X$ into $c$ classes (indexed by the 'colours'). These colourings need not satisfy any non-adjacency requirements. Given a $c$-colouring of $[X]^k$, the set of all $k$-subsets of $X$, we call a set $Y \subseteq X$ monochromatic if all the elements of $[Y]^k$ have the same colour, *i.e.* belong to the same of the $c$ partition classes of $[X]^k$. Similarly, if $G = (V, E)$ is a graph and all the edges of $H \subseteq G$ have the same colour in some colouring of $E$, we call $H$ a monochromatic subgraph of $G$, speak of a red (green, etc.) $H$ in $G$, and so on.

In the above terminology, Ramsey's theorem can be expressed as follows: for every $r$ there exists an n such that, given any $n$-set $X$, every 2-colouring of $[X]^2$ yields a monochromatic $r$-set $Y \subseteq X$. Interestingly, this assertion remains true for $c$-colourings of $[X]^k$ with arbitrary $c$ and $k$-with almost exactly the same proof!

### *Theorem 7.3: (Ramsey's Theorem, 1930)*

*For, every $r \in N$ there exists on $n \in N$ such that every graph of order at least n contains either $K^r$ or $\overline{K}^r$ as an induced subgraph.*

### *Proof*:

The asertion is trivial for $r \leq 1$.

we assume that $r \geq 2$.

Let $n: = 2^{2r-3}$, and let $G$ be a graph of order atleast $n$.

We define a sequence $V_1, V_2, \ldots, V_{2r-2}$ of sets and choose vertices $v_i \in V_i$ with the following properties;

(*i*) $|V_i| = 2^{2r-2-i}$ $\forall$ $i = 1, 2, \ldots, 2r-2$;

(*ii*) $V_i \subseteq V_{i-1} \setminus \{v_{i-1}\}$ $\forall$ $i = 2, \ldots, 2r-2$.

(*iii*) $v_{i-1}$ is adjacent either to all vertices is $V_i$ or to no vertex in $V_i$ $\forall$ $i = 2, \ldots, 2r-2$.

Let $V_i \subseteq V(G)$ be any set of $2^{2r-3}$ vertices.

Pick $v_i \in V_1$ arbitrarily.

Then (*i*) holds for $i = 1$

while (*ii*) and (*iii*) holds trivially.

Suppose now that $V_{i-1}$ and $v_{i-1} \in V_{i-1}$ have been chosen so as to satisfy (*i*) –

(*iii*) for $i-1$, where $1 < i \le 2r-2$

since $|V_{i-r} \setminus \{v_{i-1}\}| = 2^{2r-1-i} - 1$. is *odd*.

$V_{i-1}$ has a subset $V_i$ satisfying (*i*) – (*iii*);

we pick $v_i \in V_i$ arbitrarily.

Among the $(2r-3)$ vertices $v_1,\ldots, v_{2r-3}$. These are $(r-1)$ vertices that show the same behaviour when viewed as $v_{i-1}$ in (*iii*), being adjacent either to all the vertices in $V_i$ or to none. Accordingly, these $(r-1)$ vertices and $v_{2r-2}$ induce either a $K^r$ or a $\overline{K}^r$ in $G$,

because $v_1,\ldots, v_{2r-2} \in V_i\ \forall\ i$.

■

### *Theorem 7.4:*

*Let k, c be positive integers, and X an infinite set. If $[X]^k$ is coloured with c colours, then X has an infinite monochromatic subset.*

### *Proof*:

We prove the theorem by induction on $k$, with $c$ fixed. For $k = 1$ the assertion holds, so let $k > 1$ and assume the assertion for smaller values of $k$.

Let $[X]^k$ be coloured with c colours. We shall construct and infinite sequence $X_0, X_1,\ldots,$ of infinite subsets of $X$ and choose elements $x_i \in X_i$ with the following properties (for all $i$):

(*i*) $X_{i+1} \subseteq X_i \setminus \{x_i\}$;

(*ii*) all $k$-sets $\{x_i\} \cup Z$ with $Z \in [X_{i+1}]^{k-1}$ have the same colour, which we associate with $x_i$.

We start with $X_0 := X$ and pick $x_0 \in X_0$ arbitrarily. By assumption, $X_0$ is infinite. Having chosen an infinite set $X_i$ and $x_i \in X_i$ for some $i$, we $c$-colour $[X_i \setminus \{ x_i \}^{k-1}$ by giving each set $Z$ the colour of $\{ x_i \} \cup Z$ from our $c$-colouring of $[X]^k$. By the induction hypothesis, $X_i \setminus \{x_i\}$ has an infinite monochromatic subset, which we choose as $X_{i+1}$. Clearly, this choice satisfies (*i*) and (*ii*). Finally, we pick $x_{i+1} \in X_{i+1}$ arbitrarily.

Since $c$ is finite, one of the $c$ colours is associated with infinitely many $x_i$. These $x_i$ form an infinite monochromatic subset of $X$. ■

### *Lemma 7.1: (König's Infinity Lemma)*

Let $V_0$, $V_1,\ldots,$ be an infinite sequence of disjoint non-empty finite sets, and let $G$ be a graph on their union. Assume that every vertex $v$ in a set $V_n$ with $n \ge 1$ has a neighbour $f(v)$ in $V_{n-1}$. Then $G$ contains an infinite path $v_0v_1,\ldots,$ with $v_n \in V_n$ for all $n$.

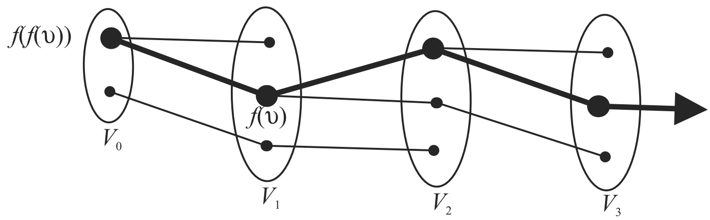



***Fig. 7.3: (Konig's Infinity Lemma)***

### *Proof*:

Let $P$ be the set of all paths of the form $v\,f(v)\,f(f(v)),\ldots,$ ending in $V_0$. Since $V_0$ is finite but $p$ is infinite, infinitely many of the paths in $P$ end at the same vertex $v_0 \in V_0$. Of these paths, infinitely many also agree on their penultimate vertex $v_1 \in V_1$, because $V_1$ is finite. Of those paths, infinitely many agree even on their vertex $v_2$ in $V_2$—and so on. Although the set of paths considered decreases from step to step, it is still infinite after any finite number of steps, so $v_n$ gets defined for every $n \in N$. By definition, each vertex $v_n$ is adjacent to $v_{n-1}$ on one of those paths, so $v_0v_1,\ldots,$ is indeed an infinite path.

### *Theorem 7.5:*

*For all k, c, r ≥ 1 there exists an n ≥ k such that every n-set X has monochromatic r-subset with respect to any c-colouring of* $[X]^k$.

### *Proof*:

As is customary in set theory, we denote by $n \in N$ (also) the set $\{0, \ldots, n-1\}$. Suppose the assertion fails for some $k$, $c$, $r$. Then for every $n \geq k$ there exist an $n$-set, without loss of generality the set $n$, and a $c$-colouring $[n]^k \to c$ such that $n$ contains no monochromatic $r$-set. Let us call such colourings bad; we are thus assuming that for every $n > k$ there exists a bad colouring of $[n]k$. Our aim is to combine these into a bad colouring of $[N]^k$, which will contradict.

For every $n \geq k$ let $V_n \neq \theta$ be the set of bad colourings of $[n]^k$. For $n > k$, the restriction $f(g)$ of any $g \neq V_n$ to $[n - 1]^k$ is still bad, and hence lies in $V_{n-1}$. By the infinity lemma, there is an infinite sequence $g_k$, $g_{k+1},\ldots,$ of bad colourings $g_n \in V_n$ such that $f(g_n) = g_{n-1}$ for all $n > k$. For every $m \geq k$, all colourings $g_n$ with $n \geq m$ agree on $[m]^k$, so for each $Y \in [N]^k$ the value of $g_n(Y)$ coincides for all $n \geq \max Y$. Let us define $g(Y)$ as this common value $g_n(Y)$. Then $g$ is a bad colouring of $[\mathrm{N}]^k$ : every $r$-set $S \subseteq \mathrm{N}$is contained in some sufficiently large n, so $S$ cannot be monochromatic since $g$ coincides on $[n]^k$ with the bad colouring $9^n$. ■

**Note**: $R(k, c, r) \to$ Ramsey Number for Parameters, $G_n$

***Proposition 7.1:***

*Let s, t be positive integers, and let T be a tree of order t. Then $R(T, K^s) = (s-1)(t-1)+1$.*

***Proof:***

The disjoint union of $(s-1)$ graphs $K^{t-1}$ contains no copy of T, while the complement of this graph, the complete $(s-1)$-partite graph $K^{s-1}_{t-1}$, does not contain $K^s$. This proves $R(T, K^s) \geq (s-1)(t-1)+1$.

Conversely, let $G$ be any graph of order $n = (s-1)(t-1)+1$ whose complement contains no $K^s$. Then $s > 1$, and in any vertex colouring of $G$ at most $S-1$ vertices can have the same colour. Hence, $\chi(G) \geq [n/(s-1)] = t$. $G$ has a subgraph $H$ with $\delta(H) \geq t-1$, which contains a copy of $T$.

## 7.4 Induced Ramsey Theorems

*For every graph $H = K^r$ there exists a graph G such that every 2-colouring of the edges of G yields a monochromatic $H \subseteq G$; as it turns out, this is witnessed by any large enough complete graph as G. Let us now change the problem slightly and ask for a graph G in which every 2-edge-colouring yields a monochromatic induced $H \subseteq G$, where H is now an arbitrary given graph.*

The fact that such a Ramsey graph exists for every choice of $H$ is one of the fundamental results of graph Ramsey theory. It was proved around 1973, independently by **Deuber, by Erdos**, **Hajnal & Posa**, and by **Rodl**.

Theorem's. **Every graph has a Ramsey graph**. In other words, for every graph $H$ there exists a graph $G$ that, for every partition $\{E_1, E_2\}$ of $E(G)$, has an induced subgraph $H$ with $E(H) \subseteq E_1$ or $E(H) \subseteq E_2$.

***Proof:***

In our construction of the desired Ramsey graph we shall repeatedly replace vertices of a graph $G = (V, E)$ already constructed by copies of another graph $H$. For a vertex set $U \subseteq V$ let $G\,[U \to H]$ denote the graph obtained from $G$ by replacing the vertices $u \in U$ with copies $H(u)$ of $H$ and joining each $H(u)$ completely to all $H(u')$ with $uu' \in E$ and to all vertices $v \in V \setminus U$ with $uv \in E$. (As shown in Fig. 7.4)

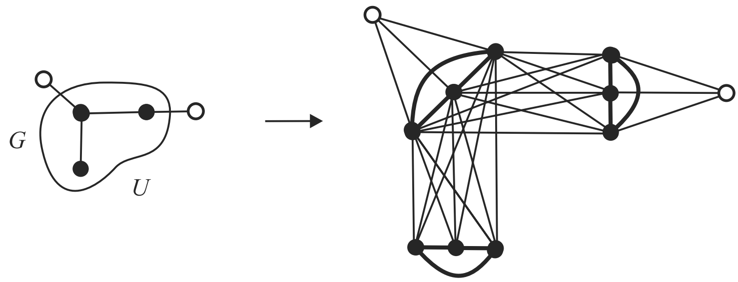



**Fig. 7.4: A Graph $G\ [U \to H]$ with $H = K^3$**

Formally, $G[U \to H]$ is the graph on

$$(U \times V(H)) \cup ((V \setminus U) \times \{\phi\})$$

in which two vertices $(v, w)$ and $(v', w')$ are adjacent if and only if either $vv' \in E$, or else $v = v' \in U$ and $ww' \in E(H)^3$.

We can prove the following:

For any two graphs $H_1$, $H_2$ there exists a graph $G = G(H_1, H_2)$ such that every edge colouring of $G$ with the colours 1 and 2 yields either an induced $H_1 \subseteq G$ with all its edges coloured 1 or an induced $H_2 \subseteq G$ with all its edges coloured 2,…,(*)

This formal strengthening makes it possible to apply induction on $|H_1| + |H_2|$, as follows:

If either $H_1$ or $H_2$ has no edges (in particular, if $|H_1| + |H_2| \leq 1$), then (*) holds with $G = \overline{K}^n$ for large enough $n$. For the induction step, we now assume that both $H_1$ and $H_2$ have at least one edge, and that (*) holds for all pairs $(H_1, H_2)$ with smaller $|H'_1| + |H'_2|$.

For each $i = 1, 2$, pick a vertex $x_i \in H_i$ that is incident with an edge. Let $H'_i := H_i - x_i$, and let $H_i''$ be the subgraph of $H'_i$ induced by the neighbours of $x_i$.

We shall construct a sequence $G^0, \ldots, G^n$ of disjoint graphs; $G^n$ will be the desired Ramsey graph $G(H_1, H_2)$. Along with the graphs $G_i$, we shall define subsets $V' \subseteq V(G')$ and a map

$$f{:}V^1 \cup, \ldots, \cup V^n \to V^0 \cup, \ldots, \cup V^{n-1}$$

such that

$$f(V') = V^{i-1} \qquad \ldots(1)$$

for all $i \geq 1$. Writing $f^i := f\,0\,, \ldots, 0\,f$ for the $i$-fold composition of $f$ whenever it is defined, and $f^0$ for the identity map on $V^0 = V(G^0)$, we thus have $f'(v) \in V^0$ for all $v \in V^i$. We call $f^i(v)$ the origin of $v$.

The subgraphs $G^i\ [V^i\ ]$ will reflect the structure of $G^0$ as follows:

*Vertices in $V^i$ with different origins are adjacent in $G'$ if and only if their origins are adjacent in $G^0$,…,(2)*

Assertion (2) will not be used formally in the proof below. However, it can help us to visualize the graphs $G^i$: every $G^i$ (more precisely, every $G^i[V^i]$–there will also be some vertices $x \in G^i - V^i$) is essentially an inflated copy of $G^0$ in which every vertex $w \in G^0$ has been replaced by the set of all vertices in $V^i$ with origin $w$, and the map $f$ links vertices with the same origin across the various $G^i$.

By the induction hypothesis, there are Ramsey graphs

$G_1 : G(H_1, H_2)$ and $G_2 : = G(H_1, H_2)$.

Let $G^0$ be a copy of $G_1$, and set $V^0 := V(G^0)$. Let $W'_0,\ldots, W'_{n-1}$ be the subsets of $V^0$ spanning an $H'_2$ in $G^0$. Thus, $n$ is defined as the number of induced copies of $H'_2$ in $G^0$, and we shall, construct a graph $G^i$ for every set $W'_{i-1}$, $i = 1,\ldots,n$. Since $H_1$ has an edge, $n \geq 1$: otherwise $G^0$ could not be a $G(H_1, H_2)$. For $i = 0,\ldots, n-1$, let $W''_i$ be the image of $V(H''_2)$ under some isomorphism $H'_2$ , $G^0[W'_i]$.

Assume now that $G^0,\ldots, G^{i-1}$ and $V^0,\ldots, V^{i-1}$ have been defined for some $i \geq 1$, and that $f$ has been defined on $V_1 \cup ,\ldots, \cup V^{i-1}$ and satisfies (1) for all $j \leq i$. We expand $G^{i-1}$ to $G^i$ in two steps. For the first step, consider the set $U^{i-1}$ of all the vertices $v \in V^{i-1}$ whose origin $f^{i-1}(v)$ lies in $W'_{i-1}$. (For $i = 1$, this gives $U^0 = W''_0$.) Expand $G^{i-1}$ to a graph $G^{i-1}$ by replacing every vertex $u \in U^{i-1}$ with a copy $G_2(u)$ of $G_2$, *i.e.* let

$$G^{i-1} : = G^{i-1}[U^{i-1} \to G_2]$$

(see Figs. 7.5 and 7.6). Set $f(u') := u$ for all $u \in U^{i-1}$ and

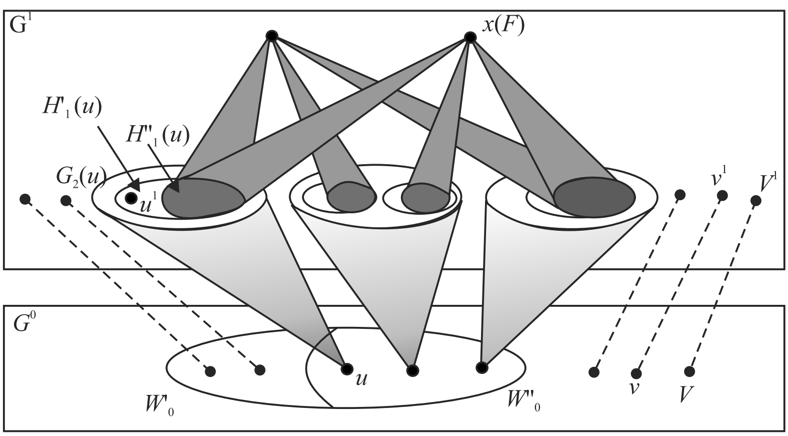



**Fig. 7.5: The Construction of $G^1$**

$u' \in G_2(u)$, and $f(v') := v$ for all $v' = (v, \phi)$ with $v \in V^{i-1} \setminus U^{i-1}$ (Recall that $(v, \phi)$ is simply the unexpanded copy of a vertex $v \in G^{i-1}$ in $G^{i-1}$.) Let $V_i$ be the set of those vertices $v'$ or $u'$ of $G^{i-1}$ for which $f$ has thus been defined, *i.e.* the

vertices that either correspond directly to a vertex $v$ in $V^{i-1}$ or else belong to an expansion $G_2(u)$ of such a vertex $u$. Then (1) holds for $i$. Also, if we assume (2) inductively for $i-1$, then (2) holds again for $i$ (in $G^{i-1}$). The graph $G^{i-1}$ is already the 'essential part' of $G^i$: the part that looks like an inflated copy of $G^0$.

In the second step we now extend $G^{i-1}$ to the desired graph $G_i$ by adding some further vertices $x \notin V^i$. Let $F$ denote the set of all families $F$ of the form

$$F = (H_1'(u) \mid u \in U^{i-1}),$$

where each $H'_1(u)$ is an induced subgraph of $G_2(u)$ isomorphic to $H'_1$. (Less formally: $F$ is the collection of ways to select from each $G_2(u)$ exactly one induced copy of $H'_1$.) For each $F \in F$, add a vertex $x(F)$ to $G^{i-1}$ and join it to all the vertices of $H''_1(u)$ for every $u \in U^{i-1}$, where $H''_1(u)$ is the image of $H''_1$ under some isomorphism $H'_1 \to H'_1(u)$. Denote the resulting graph by $G^i$. This completes the inductive definition of the graphs $G^0, \ldots, G^n$.

Let us now show that $G := G^n$ satisfies (*). To this end, we prove the following assertion (**) about $G^i$ for $i = 0, \ldots, n$:

*For every edge colouring with the colours 1 and 2, $G^i$ contains either an induced $H_1$ coloured l, or an induced $H_2$ coloured 2, or an induced subgraph $H$ coloured 2 such that $V(H) \subseteq V^i$ and the restriction of $f^i$ to $V(H)$ is an isomorphism between $H$ and $G^0[W'_k]$ for some $k \in \{i, \ldots, n-1\}$.....(*)*

Note that the third of the above cases cannot arise for $i = n$, so (**) for $n$ is equivalent to (*) with $G := G''$.

For $i = 0$, (**) follows from the choice of $G^0$ as a copy of $G_1 = G(H_1, H_2)$ and the definition of the sets $W'_k$. Now let $1 \le i \le n$, and assume (**) for smaller values of $i$.

Let an edge colouring of $G^i$ be given. For each $u \in U^{i-1}$ there is a copy of $G_2$ in $G^i$:

$$G^i \supseteq G_2(u) = G(H'_1, H_2)$$

If $G_2(u)$ contains an induced $H_2$ coloured 2 for some $u \in U^{i-1}$, we are done. If not, then every $G_2(u)$ has an induced subgraph $H'_i(u)$ $H'_i$ coloured 1. Let $F$ be the family of these graphs $H'_i(u)$, one for each $u \in U^{i-1}$, and let $x := x(F)$. If, for some $u \in U^{i-1}$, all the $x$-$H''_i(u)$ edges in $G^i$ are also coloured 1, we have an induced copy of $H_1$ in $G^i$ and are again done. We may therefore assume that each $H''_1(u)$ has a vertex $y_u$ for which the edge $xy_u$ is coloured 2. Let

$$\hat{U}^{i-1} := \{ y_u \mid \cup U^{i-1} \} \subseteq V^i.$$

Then $f$ defines an isomorphism from

$$\hat{G}^{i-1} := G^i [\, \hat{U}^{i-1} \cup \{(v, 0) \mid v \in V(G^{i-1}) \setminus U^{i-1}\}]$$

to $G^{i-1}$: just map every $y_u$ to $u$ and every $(v, \phi)$ to $v$. Our edge colouring of $G^i$ thus induces an edge colouring of $G^{i-1}$. If this colouring yields an induced $H_1 \subseteq G^{i-1}$ coloured 1 or an induced $H_2 \subseteq G^{i-1}$ coloured 2, we have these also in $G^{i-1} \subseteq G^i$ and are again home.

By (**) for $i - 1$ we may therefore assume that $G^{i-1}$ has an induced subgraph $H'$ coloured 2, with $V(H') \subseteq V^{i-1}$ and such that the restriction of $f^{i-1}$ to $V(H')$ is an isomorphism from $H'$ to $G^0[W'_k] = H'_2$ for some $k \in \{i-1,\ldots, n-1\}$. Let $H_2^1$ be the corresponding induced subgraph of $\hat{G}^{i-1} \in G^i$ (also coloured 2); then $V(\hat{H}') \in V^i$.

$f^i(V(H')) = f^{i-1}(V(H')) = W'_k$,

and $f^i$: $\hat{H} \rightarrow G^0[W'_k]$ is an isomorphism.

If $k \geq i$, this completes the proof of (**) with $H := \hat{H}'$; we therefore assume that $k < i$, and hence $k = i - 1$. By definition of $U^{i-1}$ and $G^{i-1}$, the inverse image of $W''_{i-1}$ under the isomorphism $f^i$ is $\hat{H}' \rightarrow G^0[W'_{i-1}]$ is a subset of $U^{i-1}$. Since $x$ is joined to precisely those vertices of $\hat{H}'$ that lie in $\hat{U}^{i-1}$, and all these edges $xy_u$ have colour 2, and the proof of (**) is complete.

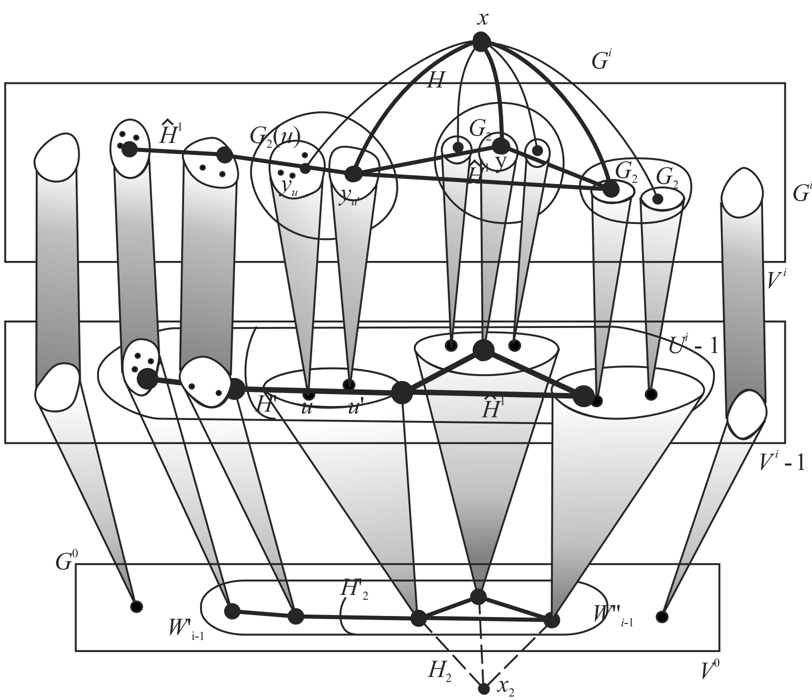



**Fig. 7.6: A Monocromatic Copy of $H_2$ in $G^i$**

## 7.5 Applications

### 7.5.1 *Schur's Theorem*

We consider the partition ({1, 4, 10, 13}, {2, 3, 11, 12}, {5, 6, 7, 8, 9} of the set of integers {1, 2,…, 13}. We observe that in no subset of the partition are there integers $x$, $y$ and $z$ (not necessarily distinct) which satisfy the equation

$$x + y = z \qquad \ldots(i)$$

Yet, no matter how we partition {1, 2,…, 14} into three subsets, there always exists a subset of the partition which contains a solution, *Schur* (1916) proved that, in general, given any positive integer $n$, there exists an integer $f_n$ such that, in any partition of $\{1, 2,\ldots, f_n\}$ into $n$ subsets, there is a subset which contains a solution. We shall show how Schur's theorem follows from the existence of the Ramsey numbers $r_n$.

***Theorem 7.7:***

*Let $(S_1, S_2,\ldots, S_n)$ be any partition of the set of integers $\{1, 2,\ldots, r_n\}$. Then, for some i, $S_i$ contains three integers x, y and z satisfying the equation $x + y = z$.*

***Proof*:**

Consider the complete graph whose vertex set is $\{1, 2,\ldots, r_n\}$. Colour the edges of this graph in colours $1, 2,\ldots, n$ by the rule that the edge $uv$ is assigned colour $j$ if and only if $|u - v| \in S_j$. By Ramsey's theorem, there exists a monochromatic triangle; that is, there are three vertices $a$, $b$ and $c$ such that $ab$, $bc$ and $ca$ have the same colour, say $i$. Assume, without loss of generality that $a > b >$ c and write $x = a - b$, y = $b - c$ and $z = a - c$. Then $x, y, z \in S_i$ and $x + y = z$.

■

***Theorem 7.8:***

*Suppose $H_1, H_2,\ldots, H_k$ are any k graphs. Then there exists an integer $R(H_1, H_2, \ldots H_k)$*, any $k$-painting of $k_v$, must contain a subgraph isomorphic to $H_i$ that is monochromatic in colour $i$ for some $i$, $1 \le i \le k$.

***Proof*:**

It is sufficient to prove the theorem in the case where all the $H_i$ are complete. Then, if $v$ is sufficiently large that a $k$-painted $K_v$ must contain a monochromatic $K_{v(H_i)}$ in color $c_i$ for some $i$, it must certainly contain a monochromatic copy of $H_i$ in color $c$, , so

$$R(H_1, H_2,\ldots, H_k) \le R(v(H_1), v(H_2),\ldots, v(H_2),\ldots, v(H_k)).$$

We proceed by induction on $k$ to prove that $R(p_1, p_2,\ldots, P_k)$ exists for all parameters. In the case $k = 2$. Now suppose it is true for $k < K$, and suppose integers $p_1, p_2,\ldots, p_k$ are given. Then $R(p_1, p_2,\ldots, p_{K-1})$ exists.

Suppose

$$v \geq R(R(p_1, p_2, \ldots, p_{k-1}), p_k).$$

Select any $k$-painting of $K_v$. Then recolor by assigning a new color $c_0$ to all edges that received colors other than $c_k$. The resulting graph must contain either a monochromatic $K_{R(p_1, p_2, \ldots, p_{k-1})}$ in color $c_0$ or a monochromatic $K_{pk}$ in color $c_K$. In the former case, the corresponding $K_{R(p_1, p_2, \ldots, p_{K-1})}$ in the original painting has edges in the $K-1$ colors $c_1, c_2, \ldots, c_{K-1}$ only, so by induction it contains a monochromatic $K_{pi}$ in color $c$, for some $i$.

***Theorem 7.9:***

*If T is a tree with m vertices, then*

$$R(T, K_n) = (m-1)(n-1) + 1$$

***Proof*:**

We can see that $R(T, K_n) > (m-1)(n-1)$. To do so, we consider a graph of $(m-1)$ disjoint copies of $K_{n-1}$ with all edges coloured red (say).

We can complete this graph to a $K_{(m-1)(n-1)}$ by colouring all remaining edges blue (say).

Since the red subgraph contains no $m$-vertex component, it contains no copy of $T$. The blue graph is $(n–1)$–partite, so it can contain no $K_n$.

Using induction on $n$. The case $n = 1$ is trivial. Suppose $n > 1$ and suppose the theorem is true of $R(T, K_s)$ whenever $s < n$. Suppose there is a coloring of the edges of in red and blue that contains neither a red $T$ nor a blue $K_n$, and examine some vertex $x$. If $x$ lies on more than $(m-1)(m-2)$ blue edges, then the subgraph of $G$ induced by the "blue" neighbors of $x$ contains either a red copy of $T$ or a blue $K_{n-1}$, by the induction hypothesis. In the former case $K_{(m-1)(n-1)+1}$ contains a red $T$, in the latter the blue $K_{n-1}$ together with $x$ forms a blue $K_n$. Therefore $x$ lies on at most $(m-1)(n-2)$ blue edges, so it lies on at least $m-1$ red edges. Since $x$ could be any vertex of the $K_{(m-1)(n-1)+1}$, the red subgraph has minimum degree at least $m-1$. Since $T$ has m – 1 edges, this red subgraph will contain a subgraph isomorphic to T. So the $K_{(m-1)(n-1)+1}$ contains a red copy of $T$, a contradiction.

### 7.5.2 Geometry Problem

The diameter of a set $S$ of points in the plane is the maximum distance between two points of $S$. It should be noted that this is a purely geometric notion and is quite unrelated to the graph-theoretic concepts of diameter and distance.

We shall discuss sets of diameter 1. A set of n points determines $\binom{n}{2}$ distances between pairs of these points. It is intuitively clear that if $n$ is 'large', then some of these distances must be 'small'. Therefore, for any $d$

between 0 and 1, we can ask how many pairs of points in a set $\{x_1, x_2, \ldots, x_n\}$ of diameter 1 can be at distance greater than $d$. Here, we shall present a solution to one special case of this problem, namely when $d = 1\sqrt{2}$.

As an illustration, consider the case $n = 6$. We then have 'six points $x_1$, $x_2$, $x_3$, $x_4$, $x_5$ and $x_6$. If we place them at the vertices of a regular hexagon so that the pairs $(x_1, x_4)$, $(x_2, x_3)$ and $(x_3, x_6)$ are at distance 1, as shown in Fig. 7.7*a*, these six points constitute a set of diameter 1.

It is easily calculated that the pairs $(x_1, x_2)$, $(x_2, x_3)$, $(x_3, x_4)$, $(x_4, x_5)$, $(x_5, x_6)$ and $(x_6, x_1)$ are at distance 1/2, and the pairs $(x_1, x_3)$, $(x_2, x_4)$, $(x_3, x_5)$, $(x_4, x_6)$, $(x_5, x_1)$ and $(x_6, x_2)$ are at distance $\sqrt{3}/2$. Since $\sqrt{3}/2 > \sqrt{2}/2 = 1/\sqrt{2}$, there are nine pairs of points at distance greater than $1/\sqrt{2}$ in this set of diameter 1.

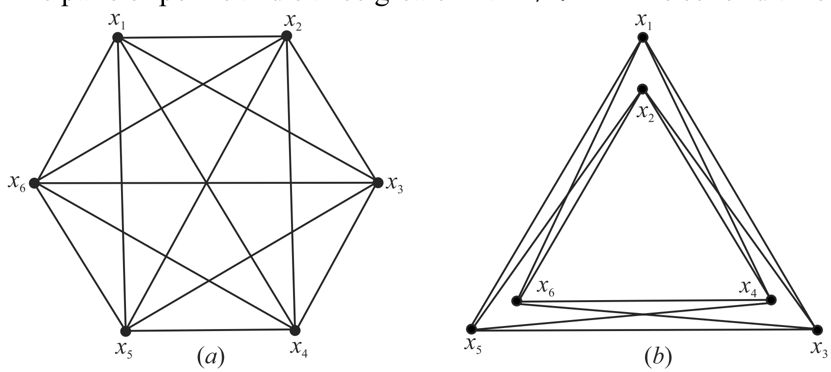



**Fig. 7.7**

However, nine is not the best that we can do with six points. By placing the points in the configuration shown in Fig. 7.7 b, all pairs of points except $(x_1, x_2)$, $(x_3, x_4)$ and $(x_5, x_6)$ are at distance greater than $1/\sqrt{2}$. Thus we have twelve pairs at distance greater than $1/\sqrt{2}$; this is, in fact, the best we can do.

### *Theorem 7.10:* (*Ramsey's Multiplicity Theorem*)

*Prove that* $N_{2,n}(K_3) = \binom{n}{3} - \left[\frac{n}{2}\left(\frac{n-1}{2}\right)^2\right]$

***Proof*:**

Suppose $K_v$ is colored in two colors, red and blue. Write $R$ for the number of red triangles, $B$ for the number of blue triangles, and $P$ for the number of partial triangles — triangles with at least one edge of each color. There are $\binom{n}{3}$ triangles in $K_v$. So

$$R + B + P = \binom{v}{3} \qquad \ldots (i)$$

Since $N_{2\cdot v}(K_3)$ equals the minimum possible value of $R + B$.

$$N_{2\cdot v}(K_3) = \binom{v}{3} - \max(P) \qquad \ldots (ii)$$

Suppose the vertices of $K_v$ are $x_1, x_2, \ldots, x_v$, and $x_i$, is incident with $r_i$, red edges. Then it is adjacent to $v - 1 - r_i$, blue edges. Therefore the $K_v$ contains. $r_i(v - 1 - r_i)$ paths of length 2 in which one edge is red and the other blue. Let us call these mixed paths. The total number of mixed paths in the $K_v$ is

$$\sum_{i=1}^{v} r_i(v-1-r_i) \qquad (iii)$$

The triangle $xyz$ can be considered as the union of the three paths $xyz$, $yzx$ and $zxy$. Moreover, the paths corresponding to different triangles will all be different. If the triangle is monochromatic, no path is mixed, but a partial triangle gives rise to two mixed paths. So there are $2P$ mixed paths in the $K_v$, and

$$P = \frac{1}{2}\sum_{i=1}^{v} r_i(v-1-r_i) \qquad (iv)$$

If $v$ is odd, the maximum value of $r_i(v - 1 - r)$ is $\frac{(v-1)^2}{4}$, attained when $r$, = $(v - 1)/2$. If $v$ is even, the maximum of $v(v - 2)/4$ is given by $r = v/2$ or $(v - 2)/2$. In either case, the maximum is

$$\left[\left(\frac{v-1}{2}\right)^2\right]$$

so

$$P \le \frac{1}{2}\sum_{L=1}^{v}\left[\left(\frac{v-1}{2}\right)^2\right]$$

$$\le \frac{v}{2}\left[\left(\frac{v-1}{2}\right)1^2\right]$$

and since $P$ is an integer,

$$\mathrm{P} \le \left[\frac{v}{2}\left[\left(\frac{v-1}{2}\right)^2\right]\right]$$

So $$N_{2,v}(K_3) \ge \binom{v}{3} - \left[\frac{v}{2}\left[\left(\frac{v-1}{2}\right)^2\right]\right] \qquad \ldots (v)$$

It remains to show that equality can be attained.

If $v$ is even, say $v = 2t$, then partition the vertices of $K_v$ into the two sets $\{x_1, x_2, \ldots, x_t\}$ and $\{x_{t+1}, x_{t+2}, \ldots, x_{2t}\}$ of size $t$, and color an edge red if it has one endpoint in each set, blue if it joins two members of the same set. (The red edges form a copy of $K_{t.t}$) Each $r_i$, equals $v/2$. If $v = 2t+1$, carry out the same

construction for $2t$ vertices, except color edge $x_i x_{i+t}$ blue for $1 \le i \le [1/2]$. Then add a final vertex $x_{2+1}$. The edges $x_i x_{2t+1}$ and $x_{i+t} x_{2t+1}$ are red when $1 \le i \le [1/2]$ and blue otherwise. In both cases it is easy to check that the number of triangles equals the required minimum. ■

***Theorem 7.11:* (*Classical Ramsey Number Theorem*)**

*For all integers p, q ≥ 2,*

$$R(p, q) \le \binom{p+q-2}{p-1}$$

***Proof*:**

We may write $n = p + q$. The proof proceeds by induction on $n$. Clearly $R(2, 2) = 2 = \binom{2+2-2}{2-1}$. Since $p, q \ge 2$, we can have $n = 4$ only if $p = q = 2$. Hence the given bound is valid for $n = 4$. Also for any value of $q$, $R(2, q) = q = \binom{2+q-2}{2-1}$ and similarly for any value of $p$, $R(p, 2) = p = \binom{p+2-2}{p-1}$, so the bound is valid if $p = 2$ or $q = 2$.

Without loss of generality we assume that $p \ge 3$, $q \ge 3$ and that

$$R(p' + q') \le \binom{p'+q'-2}{p'-1}$$

for all integers $p'$, $q'$ and $n$ satisfying $p' \ge 2$, $q' \ge 2$, $p' + q' < n$ and $n > 4$. Suppose the integers $p$ and $q$ satisfy $p + q = n$.

We apply the induction hypothesis to the case $p' = p - 1$, $q' = q$, obtaining

$$R(p-1, q) \le \binom{p+q-3}{p-2}$$

and to $p' = p$, $q' = q - 1$, obtaining

$$R(p, q-1) \le \binom{p+q-3}{p-1}$$

But by the properties of binomial coefficients.

$$\binom{p+q-3}{p-2} + \binom{p+q-3}{p-1} = \binom{p+q-2}{p-1}$$

and from Lemma, we have

$$R(p, q) \le R(p-1, q) + R(p, q-1).$$

So, $$R(p, q) \le \binom{p+q-3}{p-2} + \binom{p+q-3}{p-1} = \binom{p+q-2}{p-1}$$

■

## SUMMARY

1. Ramsey theory is the study of partitions of large structures.
2. Ramsey theorem is the generalization of pigeonhole principle as partitions of sets (as substructures).
3. It T is a spanring tree of the $k$-dimensional cube $Q_k$, then there is an edge of $Q_k$ outside $T$ whose addition to $T$ creates a cycle of length at least $2k$.
4. Every list of more than $n^2$ distinct numbers has a monotone sublist of length more than $n$.
5. If $\Sigma\ p_i - k + 1$ object are partitioned into k-classes with quotes $\{p_i\}$, then some class must meet its quota.
6. Like the pigonhole principle, Ramsey's theorem has subtle and fascinating applications.
7. Ramsey's theorem defines the Ramsey numbers $R(p_1, p_2, ...., p_k; r)$. No exact formula is known, and few Ramsey numbers are to be computed.
8. Ramsey's theorem for $r = 2$ says that k-coloring the edges of a large enough complete graph forces a monochromatic complete subgraph.
9. R(G, G) is called the Ramsey number of G for all 113 graph with at most six edges and no isolated vertices.
10. If T is an m-vertex tree, then $R(T, K_n) = (m - 1)(n - 1) + 1$.
11. A completely labeled cell is a cell having all three labels on its corners.

## EXERCISES

1. A graph $G$ is $b$-critical if $b(G - e) < \beta(G)$ for all e $\in$ $E$. Show that
   (*a*) A connected β-critical graph has no cut vertices;
   (*b*) If $G$ is connected, then $\beta\ \frac{1}{2}(\in + 1)$.
2. For any two integers $k$ ³ 2 and $l$ ³ 2, Prove that, $r\ (k, l)$ £ $r(k, l - 1) + r(k - 1, l)$.
3. Prove that, $r(k, l) \geq \binom{k+l-2}{k-1}$
4. Prove that, $r\ (k, k) \geq 2^{k/2}$.
5. Show that for all $k$ and $l$, $r\ (k, l) = r\ (l, k)$.
6. Let $r_n$ denote the Ramsey number $r(k_1, k_2,\ldots, k_n)$ with $k_i = 3$ for all $i$.
   (*a*) Show that $r_n \leq n(r_{n-1} - 1) + 2$.
   (*b*) Noting that $r_2 = 6$, use (*a*) to show that $r_n \leq [n!\ e] + 1$.
   (*c*) Deduce that $r_3 \leq 17$.

7. The composition of simple graphs $G$ and $H$ is the simple graph $G[H]$ with vertex set $V(G) \times V(H)$, in which $(u, v)$ is adjacent to $(u', v')$ if and only if either $uu' \in E(G)$ or $u = u'$ and $vv'$ $E(H)$.
   (*a*) Show that $a(G[H]) \leq a(G)$ $a(H)$.
   (*b*) Using (*a*), show that
   $r(kl + 1, kl + 1) - 1 \geq (r(k + 1, k + 1) - 1) \times (r(l + 1, 1 + l) - 1)$
   (*c*) Deduce that $r(2^n + 1, 2^n + 1) \geq 5^n + 1$ for all $n \geq 0$.
8. Let $G_1, G_2, \ldots, G_m$, be simple graphs. The generalised Ramsey number $r(G_1, G_2, \ldots, G_m)$ is the smallest integer $n$ such that every m-edge colouring $(E_1, E_2, \ldots, E_m)$ of $K_n$ contains, for some $i$, a subgraph isomorphic to $Gi$ in colour $i$. Show that
   (*a*) If $G$ is a path of length three and $H$ is a 4-cycle, then $r(G, G) = 5$, $r(G, H) = 5$ and $r(H, H) = 6$;
   (*b*) If $T$ is any tree on $m$ vertices and if $m - 1$ divides $n - 1$, then $r(T, K_{1,n}) = m + n - 1$;
   (*c*) If $T$ is any tree on $m$ vertices, then $r(T, K_n) = (m - 1)(n - 1) + 1$.

   A certain bridge club has a special rule to the effect that four members may play together only if no two of them have previously partnered one another. At one meeting fourteen members, each of whom has previously partnered five others, turn up. Three games are played, and then proceedings come to a halt because of the club rule. Just as the members are preparing to leave, a new member, unknown to any of them, arrives. Show that at least one more game can now be played.
10. Show that $s_1 = 2$, $s_2 = 5$ and $s_3 = 14$.

## Suggested Readings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❑❑❑

# 8 Enumeration and Pölya's Theorem

George Pölya
(1887-1985)

George Pölya was born on December 13, 1887 at Budapestm Austria Hungary. He became a professor of mathematics in 1914 at ETH Zurich in Suritzerland and worked up to 1940. Later on he shifted to Stanford University and worked upto 1953. He remained stanford professor Emeritus for the rest of his life and career. He worked on a range of mathematical topics, including series, number theory, mathematical analysis, geometry, algebra, combinatorics and probability theory. He is known for multivariate Pölya distribution, Pölya conjucture, Pölya enumeration theorem, Landau–Kolmogorov inequality, Pölya–Vinogradov inequality, Pölya ineqality, Pölya aeppli distribution and Pölya urn model. He died in Polo Alto, California (USA) on Setember 7, 1985 (aged 97)

## 8.1 Introduction

The famous mathematician Arthur Cayley (1857), became interested in graph theory for the purpose of counting trees. The number of trees of n–vertices gave him the number of isomers of the saturated hydrocarbon with $n$ carbon atoms $(C_nH_{2n+1})$. This work on counting of different types of graphs, and the results been applied in solving some practical as enumeration.

We can not end this theme without considering the most basic question about graphs to count the graphs with a given set of vertices or we may be interested in the number of isomorphic classes of certain graphs.

All graph enumeration problems have two categories viz.

**Category 1.** Counting the number of different graphs of a particular type (non isomorphic)


S. K. Yadav, *Advanced Graph Theory*,
https://doi.org/10.1007/978-3-031-22562-8_8

**Category 2**. Counting the number of sub graphs of a particular kind in a graph G, such as the number of edge–disjoint paths of lengths $k$ between vertices $a$, $b$ in $G$ (i.e. matrix representation)

## 8.2 Labelled Counting

Let $n \in N$ and $P$ be a graph-theoretic property. A labelled counting or balled enumeration is a counting of the number of graphs on the vertices $\{u_1, u_2, \ldots, u_n\}$ That satisfy the property $P$.

Let $G$ and $G'$ be the graphs on the vertices $\{u_1, u_2, \ldots, u_n\}$ That both satisfy the property $P$. In a labelled counting $G$ and $G'$ are the same iff every two vertices $u_i$ and $u_j$ are joined by the same number of edges in both graphs.

### Note:

(*i*) The property $P$ may be any property which can be stated in graph theoretic terms: being simple, connected, Eulerian, Hamiltonian, 5–colorable, planer, acylic, having perfect matching, containing exactly 2004 edges and so or any combination of much graph theoretic properties.

(*ii*) Two graphs $G$ and $G'$ on vertices $\{u_1, u_2, \ldots, u_n\}$ are the same in a labelled counting (enumeration), if the edge multiplicity of each pair of vertices in $G$ is the same as $G'$. (the edges are not labeled at all).

■ **Example 8.1:** *Labelled graph of three vertices*

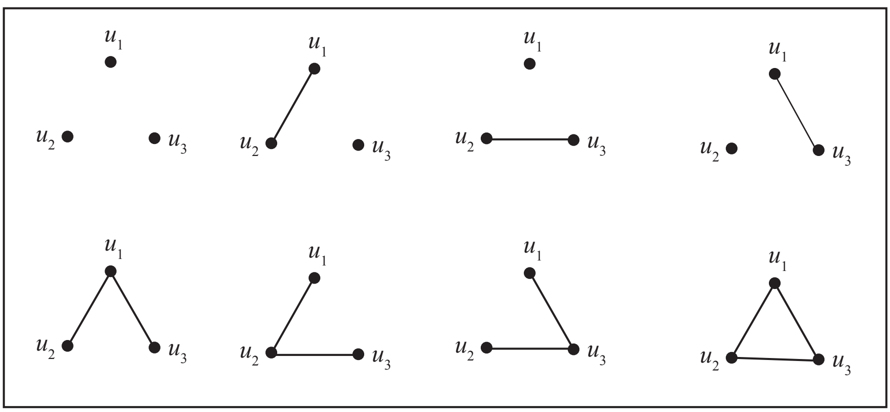



**Fig. 8.1: Eight labeled simple graphs on three vertices**

We want to count the number of labelled simple graphs on three vertices.

Counting the number of simple graphs on the vertices

$V = \{u_1, u_2, \ldots, u_n\}$

$p$ is simple.

for every pair $u_i$ and $u_j$ disjoint vertices from V, either they are adjacent or not.

There are $\binom{3}{2} = 3$ distinct pairs of vertices in $V$ *viz* $\{u_1, u_2\}$ $\{u_2, u_3\}$ and $\{u_1, u_3\}$

Hence there are precirely $2\binom{3}{2} = 8$ possibilities of forming a simple graph on $V = \{u_1, u_2, u_3\}$

Hence there are 8 labelled simple graphs on three vertices.

## 8.3 Unlabelled Counting

Let $n \in N$ and $P$ be fa graph theoretic property. An unlabeled counting or an unlabelled enumeration is a counting of the number of graph on the vertices $(u_1, u_2, \ldots, u_n\}$ that satisfy the property P and are not isomorphic.

Let $G$ and $G'$ are the graph on the vertices $(u_1, u_2, \ldots, u_n\}$ that both satisfy the property $P$. In an unlabelled counting, $G$ and $G'$ are considered to be the same iff they are isomorphic i.e. $G \cong G'$.

■ **Example 8.2**: *Number of unlabelled simple graphs on three vertices.*

All simple graphs with one edgs are isomorphic, as are all simple graphs with two edges. No other pairs of labelled simple graphs on 3 labelled vertices V = $\{v_1, v_2, v_3\}$ are isomorphic. Hence the number of unlabelled single graphs on $n$ = 3 vertices in four as shown in Fig 8.2.

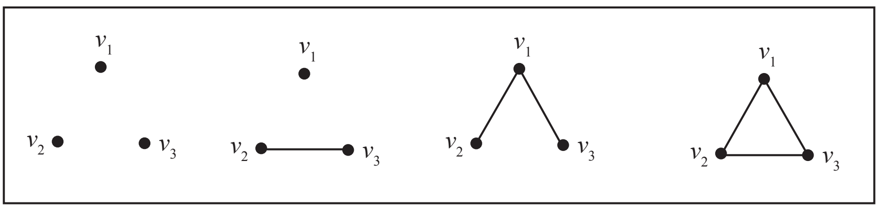



**Fig. 8.2: There are only four nonisomorphic simple graphs on three vertices.**

### *Theorem 8.1*

*Then number of simple, labelled graphs of n vertices is* $2^{n(n-1)/2}$

### Proof:

The number of simple graphs of $n$ vertices and 0, 1, 2,…, $n(n-1)/2$ for $e$. (There are $n(n-1)/2$ unordered pairs of vertices. If we regard the vertices as distinguishable from one another (labelled graph).

There are $\left(\begin{matrix} \frac{n(n-1)}{2} \\ e \end{matrix}\right)$ ways of selecting $e$ edges to from the graphs.

The sum of all such numbers is the number of all simple graph with $n$ vertices.

We can use the following identity to prove the theorem.

$$\binom{K}{0}+\binom{K}{1}+\binom{K}{2}+,\ldots,+\binom{K}{k-1}+\binom{K}{k}=2^k$$

### *Theorem 8.2*

*There are $n^{n-2}$ trees on n labelled vertices.*

### Proof:

Let V = $\{1, 2,\ldots, n\}$be the set of vertices.

Given a tree T, associate a code with T.

Remove the and vertex with the smallest label and write down the label of the adjacent vertex. Repeat the process until only two vertices remain. The code obtained is a sequence of length $(n-2)$ consisting of some numbers from 1, 2,…, $n$ any number may occur several times code in the code (As shown in Fig 8.3. Each of the $n^{n-2}$ possible codes correspond to a unique tree.

### Corollary 1.

Let $d_1 \leq d_2 \leq ,\ldots, \leq d_n$ be the degree sequence of a tree. $d_1 \geq 1$ and $\sum_{i=1}^{n} d_i = 2n-2s$ . Then the number of labelled trees of order n with degree sequence $(d_i)_1^n$ given by the multinomial coefficient.

$$\binom{n-2}{d_1-1,\ \ d_2-1,\ \ldots,\ d_n-1}$$

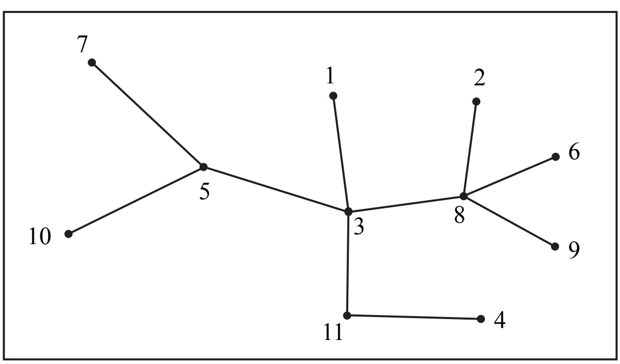



**Fig. 8.3 : The Priefer code of this tree is (3, 8, 11, 8, 5, 8, 3, 5, 3)**

■

## 8.4 Generating Function

Generating functions are the tools which connect discrete mathematics to continuous analytic mathematics. In this matter we include additional tools of our tool box to treat discrete object i.e. the known methods and techniques from analytic mathematics. These tools have proved invaluable results for solving many problems of combinatorics. Some minimal knowledge is needed in order to understand generating function from analytic mathematics. The purpose of generating functions is two fold viz.

(*i*) To use known analytic methods an a particular functions, $f(x)$ or $f(x,y)$, to obtain the corresponding closed expression or formula for the number of possibilities of a certain finite combinational structure.

(*ii*) The cryptological aspect i.e. to encode the number of possibilities of a certain finite combinatorio structure in terms of simple analytic function, $f(x)$ or $f(x,y)$. Since very often it is possible to state the discrete results of functions where it is not possible to do so in a discrete fashion. On decoding the function, many times we can obtain just enough information for the desired purpose.

Let $\{a_0, a_1, a_2,\ldots, a_n\} = \{a_n\}_{n\geq 0}$ be a given sequence of real or complex number then.

1. The generating function $a(x)$ of the sequence $\{a_n\}_{n\geq 0}$ is given by a series.
   $a(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + ,\ldots, a_n x^n = \sum_{n\geq 0} a_n x^n$ ($x$ be a dummy variable)
2. The exponential generating function $a(x)$ of the sequence $\{a_n\}_{n\geq 0}$ is given by the series.

$$a(x) = a_0\frac{1}{0!} + a_1\frac{x}{1!} + a_2\frac{x^2}{2!} +,\ldots, a_n\frac{x^n}{n!} = \sum_{n\geq 0} a_n\frac{x^n}{n!} \quad (x \text{ be a dummy variable})$$

### For example

(*i*) The generating functions for a constant sequence (1, 1, 1, …, 1) consisting entirely of 1's, in $1+ x + x^2 + ,\ldots, + x^n = \sum_{n\geq 0} x^n = \frac{1}{1-x}$

(*ii*) The generating function for the sequence $\left(1, 1, \frac{1}{2}, \frac{1}{6}, \ldots, \frac{1}{n!}\right)$ is $\left(1, 1, \frac{1}{2}, \frac{1}{6}, \ldots, \frac{1}{n!}\right) = e^x$ (the exponential function)

(*iii*) The generating function for the sequence (1, 1, 2, 6 ,…, n!,…) is $1 + x + 2x^2 + 6x^3 + ,\ldots, n!\, x^n = \sum_{n\geq 0} n!x^n$

***Note:*** The above example are valid only for real and complex numbers.

■ **Example 8.3:** *Consider the sequence of trees $T_1$, $T_2$,..., as shown in Fig. 8.4. Let n > 1 and all trees on n or fewer vertices where all of the vertices have degree three or less. Obtain generating function by using enumeration of graphs.*

## Solution

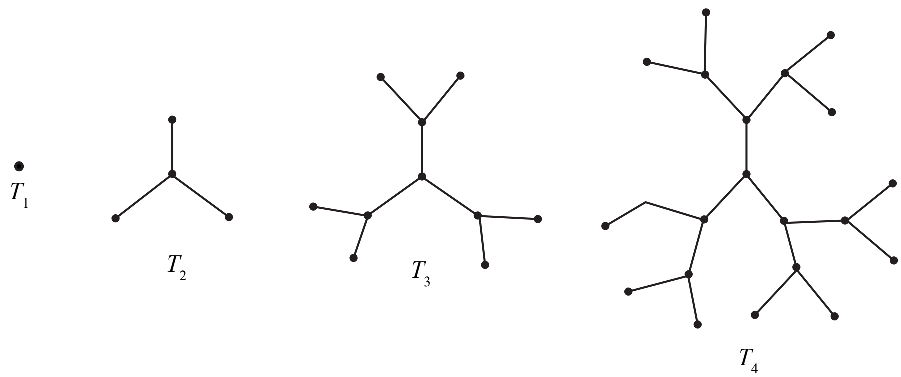



**Fig. 8.4: The first four trees in the sequence $T_1$, $T_2$,..., $T_k$**

Let $t_k$ denotes the number of vertices of the $k$th tree $T_k$ and satisfies.

$$t_1 = 1, t_2 = 4$$
$$t_k = 3t_{k-1} - 2t_{k-1} \text{ for all } k \geq 3$$

Let $a_k = t_{k-1}$, we can derive a closed formula for $a_k$ and have for $t_k$ as well. We consider generating functions:

$$a(x) = a_0 + a_1x + a_2x^2 + ,\ldots, + a_kx^k = \sum_{k\geq 0} a_kx^k$$

Since $a_0 = 1$, $a_1 = 4$ and $a_k = 3a_{k-1} - 2a_{k-2}$ $\forall\ k \geq 2$.

For the generating function, we get

$$\begin{aligned} a(x) &= a_0 + a_1x + \sum_{k\geq 0} a_kx^k \\ &= 1 + 4x + \sum_{k\geq 0} (3a_{k-1} - 2a_{k-2})\, x^k \\ &= 1 + 4x + 3x\left(\sum_{k\geq 2} a_{k-1}\, x^{k-1}\right) + 2x^2\left(\sum_{k\geq 2} a_{k-2}\, x^{k-2}\right) \\ &= 1 + 4x + 3x\,[a(x) - 1] + 2x^2\, a(x) \end{aligned}$$

Solving for $a(x)$ and using partial fractions, we get

$$a(x) = \frac{1+x}{1-3x+2x^2} = \frac{3}{1-2x} - \frac{2}{1-x}$$

Expanding the last two functions into Geometric series, we get

$$a(x) = 3\sum_{k\geq 0} 2^k x^k - 2\sum_{k\geq 0} x^k$$

Since $a_k$ is the coefficient of $x^k$ in $a(x)$.

It must be the coefficient of $x^k$ on the right hand side.

We obtained that

$$a_k = 3.\ 2^k - 2$$

Hence $$t_k = 3.\ 2^{k-1} - 2$$

(Derived formula for $t_k$)

■ **Example 8.4 :** *Define the catalan number $C_0, C_1, C_2, \ldots,$ recursively.*

### Solution

The number $\{C_n : n \geq 0\} = \{C_0, C_1, C_2, \ldots\}$ defined recursively by $C_0 = 1$.

$$C_1 = 1$$

and $$C_n = \sum_{i=0}^{n-1} C_i . C_{n-i-1} \qquad \text{for } n \geq 2.$$

are called Catalan numbers. Hence $C_n$ is called the nth Catalan number.

$\because$ $$C_n = C_0\, C_{n-1} + C_1\, C_{n-2} + ,\ldots, C_{n-1}\, C_0$$

Let $C(x)$ be the generating function for the Catalan numbers.

$$C(x) = C_0 + C_1 x + C_2 x^2 + ,\ldots, = \sum_{n\geq 0} C_n x^n \qquad \ldots (i)$$

From equ. $(i)$ we can see that

$C(x) - C(x) = C(x)^2$ is given by

$C(x)^2 = C_0^2 + (C_0\, C_1 + C_1\, C_0)\, x + (C_0\, C_2 + C_1\, C_1 + C_0\, C_2)\, x^2 + ,\ldots,$

$$= \sum_{n\geq 1}\left(\sum_{i=0}^{n-1} C_i\, C_{n-i-1}\right) x^{n-1} \qquad \ldots (ii)$$

Multiplying both side by $x$ in equ. $(ii)$, we get the sequence definition of $C_n$ as.

$$x\, C(x)^2 = \sum_{n\geq 1}\left(\sum_{i=0}^{n-1} C_i\, C_{n-i-1}\right) x^n$$

$$= \sum_{n\geq 1} C_n\, x^n = C(x) - 1$$

Solving for $C(x)$ at $x = 0$ yields $C(0) = C_0 = 1$

i.e, $$C(x) = \frac{1}{2x}\left(1 - \sqrt{1-4x}\right) \qquad \ldots (iii)$$

using generalized Binomial theorem,

$$(1+x)^\alpha = \sum_{n\geq 0}\binom{\alpha}{n}x^n \text{ for any real } \alpha.$$

Where $$\binom{\alpha}{n} = \alpha\,(\alpha-1)\,,\ldots,\,(\alpha-1+1)/n!$$

$\therefore$ $$C(x) = \frac{1}{2x}\left(1-\sqrt{(1-4x)}\right)$$

$$= -\frac{1}{2x}\sum_{n\geq 0}\binom{\frac{1}{2}}{n}(-4x)^n$$

$$= \sum_{n\geq 0}\frac{1}{n}\binom{2n-2}{n-1}x^{n-1} \qquad \ldots (iv)$$

Comparing coefficients for $x^{n-1}$, we get

$$C_{n-1} = \frac{1}{n}\binom{2n-2}{n-1} \quad \forall\, n\geq 1$$

$\therefore$ $$C_n = \frac{1}{n+1}\binom{2n}{n} \quad \forall\, n\geq 0$$

## 8.5 Partitions of a Finite Set

In graphs and hyper graphs, the presentation of a fundamental labelled counting lemma for graphs depends on the use of exponential generating functions and the use of weak ordered partitions of a set.

Let $K \in N$ and $S$ be a fixed finite set containing $n \geq k$ elements then.

(*i*) A *weak ordered partition* of s into k parts is an ordered $k$ – tuple $\{S_1, S_2, \ldots, S_k)$ of subsets of $S$ satisfying:

$$S_i \cap S_j = \phi\,,\ i.e.\ \bigcup_{i=1}^{k} S_i = \mathrm{S} \qquad \forall\ i\neq j$$

(*ii*) A *proper partition* of $S$ into $k$ parts is a $k$–elements set $\{S_1, S_2, \ldots, S_k\}$ of none empty subsets of S satisfying the condition in section (*i*)

The above definition can be illustrated as under

me consider the set $S = \{1, 2, 3, 4\}$

The following are all the weak ordered partitions of $S$ into two parts $(S_1, S_2)$ :

($\phi$ { 1, 2, 3, 4}) , ({1}, { 2, 3, 4}), ({2} { 1, 3, 4}), ({3} { 1, 2, 4}), ({4} {1, 2, 3}), ({1, 2} {3, 4}) , ({2, 3} {1, 4}), ({3, 4} {1,2}), ({1, 3} {2, 4}), ({2, 4} {1, 3}), ({2, 4}, {1, 3}), ({1, 4} {2, 3}), ({1, 2, 3} {4}) ({1, 2, 4} { 3}), ({1, 3, 4} {2}), ({2, 3, 4} {1}), ({1, 2, 3, 4}$\phi$)

Since ({1} (2, 3, 4}) and ({2, 3, 4} {1}) are distinct weak ordered partitions.

Since there are $2^4$ subsets of (1, 2, 3, 4)

When ever we consider proper partitions of $S$ into nonempty parts $\{S_1, S_2\}$, there is no distinction between $\{S_1, S_2\}$ and $\{S_2, S_1\}$

Hence there are precisely seven proper partitions of S into two nonempty parts $\{S_1, S_2\}$:

({1} {2, 3, 4}), ({2} {1, 3, 4}), ({3} {1, 2, 4}), ({4} {1, 2, 3}), ({1, 2} {3, 4}), ({2, 3} {1, 4}), ({1, 3} {2, 4})

**Note:**

The number of weak ordered partitions of $S$ into two proper parts is given by.

$2!\ S(4, 2) + 2$ Where $S(4, 2) = 7$

i.e. the number of proper partitions of S into two nonempty parts.

## Theorem 8.3

*The stirling number of second kind satisfy the following recursive equation:*

$S(n, k) = K\ S(n-1, k) + S(n-1, k-1)$ for all $k, n \in \mathrm{N}$

## Proof:

There are two types of partitions of the set $\{1, 2, \ldots, n\}$ into $k$ parts.

(*i*) The first type consists of those partitions where the set containing $n$ is singleton set $\{n\}$

In this case the remaining $(k-1)$ sets form a nonempty partition of $\{1, 2, \ldots, n-1\}$

Hence, there are $S(n-1, k-1)$ partitions of the first type

(*ii*) The second type consists of those nonempty partition where the set containing $n$ has two or more elements in it.

On removing $n$ from the set yields one of the set $S(n-1, k)$ partitions of $\{1, 2, \ldots, n)$ into k parts.

Since $n$ could have been in any of the $k$ possible sets, we can see that there are $k.\ S(n-1, k)$ partitions of the second type.

Which completes the entire proof. ∎

**Note:**

The Stirling numbers of the second kind are named after the scoltish mathematician **James Stirling** (1692 –1770)

### Theorem 8.4

The Bell numbers $B(n)$, where n is a non–negative integer, satisfy the recessive formula

$$B\,(n+1) = \binom{n}{0} B(0) + \binom{n}{1} B(1) + ,\ldots, + \binom{n}{n} B(n)$$

$$= \sum_{k=0}^{n} \binom{n}{k} B(k)$$

### Proof:

For $n \in$ N

The unrestricted nonempty partitions of $\{1,\ldots, n, n+1\}$ are of exactly $(n+1)$ is contained in a set with $k$ elements from $\{1,\ldots, n\}$

There are $\binom{n}{k}$ ways of choosing $k$ elements to join $(n+1)$ in a set and are $B\,(n-k)$ unrestricted partitions of the remaining $(n-k)$ elements]

Since $\binom{n}{k} = \binom{n}{n-k}$

We have

$$B\,(n+1) = \sum_{k=0}^{n} \binom{n}{k} B\,(n-k)$$

$$= \sum_{k=0}^{n} \binom{n}{k} B\,(k)$$

Hence the result. ■

## 8.6 The Labelled counting Lemma

**This lemma is fundamental for counting labelled graphs.**

For a non negative $n$, let $a_0$ be the number of labelled graphs on $n$ vertices satisfying the graphs property $P$(a) and let $b_n$ be the number of labelled graphs on $n$ vertices satisfying the graphs property $P\,(b)$.

Let $\bar{a}\,(x)$ and $\bar{b}\,(x)$ be the corresponding exponential generating function for the sequences $\{a_n\}_{n \geq 0}$ and $\{b_n\}_{n \geq 0}$ respectively.

If $C_n$ is defined by

$$C_n = \sum_{(S_1\, S_2)} a_{|s_1|} \quad b_{|s_2|}$$

Where the sum is taken over all weak ordered partitions $(S_1, S_2)$ of $\{1,\ldots, n\}$, then the exponential generating functions $\bar{c}\,(x)$ of the sequence $\{C_n\}_{\mathrm{n} \geq 0}$ is given by.

$$\bar{c}\,(x) = \bar{a}\,(x)\, \bar{b}(x)$$

**Proof:**

For each $n \in N$ and $i \in \{a, 1, \ldots, n\}$

There are $\binom{n}{i}$ weak ordered partitions $(S_1, S_2)$ of $\{1, 2, \ldots, n\}$ where $|S_1| = i$ and $|S_2| = n - i$

Hence we have

$$C_n = \sum_{i=0}^{n} \binom{n}{i} a_i \, b_{n-i}$$

Which in the coefficient for

$\frac{x^n}{n!}$ in $\bar{a}(x)\,\bar{b}(x)$ ■

## 8.7 Permutations

Let $X$ be a set. By a permutation on $X$ we mean a bijection mapping $\sigma : X \to X$

We only study permutations in the case when the set $X$ is finite.

A permutation on $X = \{a_1, a_2, \ldots, a_m\}$ will be denoted by a $2xm$ matrix.

i.e. $$\sigma = \begin{pmatrix} a_1 & a_2 & \ldots & a_m \\ \sigma(a_1) & \sigma(a_2) & \ldots & \sigma(a_m) \end{pmatrix}$$

Since $\sigma$ is bijective, each element on $X$ is mapping onto exactly one element in $X$ by $\sigma$. Hence each element in $X$ occurs exactly once in the second row of the matrix shown above.

**Note:** Let $\rho$ and $\sigma$ be permutations on $X$. Then the product $\rho\sigma$ of $\rho$ and $\sigma$ is defined as the composite mapping $\rho o \sigma$.

For each $x \in X$, we have

$$\rho\sigma(x) = \rho o \sigma(x) = \rho(\sigma(x)).$$

***Illustration:***

Let $\rho = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$ and $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$

Then
$$\rho\,\sigma\,(1) = \rho\,(\sigma\,(1)) = \rho\,(3) = 2$$
$$\rho\,\sigma\,(2) = \rho\,(\sigma\,(2)) = \rho\,(2) = 3$$
$$\rho\,\sigma\,(3) = \rho\,(\sigma\,(3)) = \rho\,(4) = 1$$
$$\rho\,\sigma\,(4) = \rho\,(\sigma\,(4)) = \rho\,(1) = 1$$

Hence

$$\rho\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$$

### 8.7.1 Cycle Index

The cycle index of a permutation group $G$ is the average of

$$a_1^{j_1(g)}\, a_2^{j_2(g)}\, a_3^{j_3(g)} \dots$$

Over all permutation $g$ of the group, where $j_k(g)$ is the number of cycles of length $k$ in the disjoint cycle decomposition of $g$.

The cycle index $Z(G)$ of $G$ is given by

$$\mathrm{Z(G)} = \frac{1}{|G|} \sum_{g \in G} \prod_{k=1}^{n} a_k^{j_k(g)}$$

## 8.8 Pölya's Enumeration Theorem

Polya's therorem can be used to enumerate objects under permutation groups. By using group theory, combinations and some other example Pölya's theorem and Burnside's lemma are derived. The example used in Pöolya's theorem are a square, pentagon, hexagon and heptagon under their respective dihedral graphs (group), generalization by using more permutation and applications to graph theory. Using Pölya's theorem, Harary and Palmer give a functions which gives the number of unlabelled graphs of $n$ verticer, and $m$ edges.

Suppose $G$ is a group of permutations of a set $X$, and let $\overline{G}$ be the induced group of permutations of the set $\Psi$ of colour of $X$. Now each permutation $g$ in $G$ induces a permutation $\hat{g}$ of $\Psi$ in the following way. Given a colouring ω, we define $\hat{g}(\omega)$ to be the colouring in which the colour assigned to x is the colour ω assigns to $g(x)$; that is.

$$((\hat{g})(\omega))(x) = \omega(g(x))$$

We require the generating function $K_E(c_1, c_2, \dots, c_k)$, where $E$ is a set of colouring containing one representative of each orbit of $\overline{G}$ on $\Psi$. The coefficient of $c_1^s c_2^j, \dots$, in $K_E$ will be the number of distinguishable colouring in which colour $c_1$ is used $x$ times, colour $c_2$ is used t times, and so on.

Pölya's theorem state that $K_E$ is obtained from the cycle index $Z_G(a_1, a_2, \dots a_n)$ by substituting

$$c_1^i + c_2^i + , \dots, + c_k^i$$

for $a_i (1 < i < n)$. Before going to the proof, let us see how this works in the simple case of the red-and-white colouring of the corners of a square.

Let $G$ be a group of permutations of a set $X$, where frequently we take $X$ to be the set $\{1, 2, \dots, n\}$. Each element $g$ in $G$ can be written in cycle notation with $j_i$ cycles of length $i (1 \le i \le n)$, and we recall that the type of $g$ is the corresponding partition.

$$[1^{j_1} 2^{j_2}, \dots, n^{j_n}]$$

of $n$. We have $j_1 + 2j_2 + ,\ldots, + nj_n = n$ . We shall associate with g an expression

$$Z_g\,(a_1, a_2, \ldots, a_n) = a_1^{j1}\, a_2^{j2}\, ,\ldots,\, a_n^{jn}$$

where the $a_i(1 < i < n)$ are, for the moment, simply formal symbols like the $x$ in a polynomial. For example, if $G$ is the group of symmetries of a square, regarded as permutations of the corners 1, 2, 3, 4 then the expression $Z_g$ are given in Table 8.1. Note that although the 1-cycles are conventionally omitted in the notation for $g$ it is important to include them in $Z_g$.

**Table 8.1:** Cycles in a sequence

| $g$ | $j_1$ | $j_2$ | $j_3$ | $j_4$ | $Z_g$ |
|---|---|---|---|---|---|
| $id$ | 4 | – | – | – | $a_1^4$ |
| (1234) | – | – | – | 1 | $a_4$ |
| (13)(24) | – | 2 | – | – | $a_2^2$ |
| (1432) | – | – | – | 1 | $a_4$ |
| (12)(34) | – | 2 | – | – | $a_2^2$ |
| (14)(23) | – | 2 | – | – | $a_2^2$ |
| (13) | 2 | 1 | – | – | $a_1^2 a_2$ |
| (24) | 2 | 1 | – | – | $a_1^2 a_2$ |

The formal sum of the $Z_g$ taken over all $g$ in $G$, is a 'polynomial in $a_1$, $a_2,\ldots, a_n$. Dividing by $|G|$ we obtain the cycle index of the group of the group of permutations:

$$Z_g\,(a_1, a_2,\ldots, a_n) = \frac{1}{|G|}\sum_{g \in G} Z_g(a_1, a_2,\ldots,a_n)$$

For example, the cycle index of the group of the square is, as considered above, is

$$\frac{1}{8}\left(a_1^4 + 2a_1^2 a_2 + 3a_2^2 + 2a_4\right)$$

and we have to substitute

$$a_1 = r + w,\quad a_2 = r^2 + w^2,\quad a_3 = r^3 + w^3,\quad a_4 = r^4 + w^4$$

we get

$$K_E(r, w) = \frac{1}{8}\left[(r+w)^4 + 2(r+w)^2\left(r^2+w^2\right) + 3\left(r^2+w^2\right)^2 + 2\left(r^4+w^4\right)\right]$$
$$= r^4 + r^3w + 2r^2w^2 + rw^3 + w^4$$

**Theorem 8.5**

*The number of orbits of G on X is.*

$$\frac{1}{|G|}\sum_{g \in G}|F(g)| \text{ where } F(g) = \{\, x \in X, g \in G \mid g(x) = x\}$$

Its weighted form is

$$\sum_{x \in D} I(x) = \frac{1}{|G|} \sum_{g \in G} \sum_{x \in F(g)} I(x)$$

**Proof:**

Let $G$ be a group of permutations of $X$, and let $I(x)$ be an expression which is constant on each orbit of $G$, so that

$$I(g(x)) = I(x) \text{ for all } g \in \underline{G}, x \in X$$

Let $D$ be a set of representatives, one from each orbit, and let $E = \{(g, x)|g(x) = x\}$. By evaluating the sum

$$\sum_{(g,x) \in E} I(x)$$

in two different ways, we can show that

$$\sum_{x \in D} I(x) = \frac{1}{|G|} \sum_{g \in G} \sum_{x \in F(g)} I(x)$$

This is the required 'weighted' version. ■

**Theorem 8.6 (Polya)**

*Let $Z_G(a_1, a_2, ..., a_n)$ be the cycle index for a group G of permutations of X. The generating function $K_E(c_1, c_2, ..., c_n)$ for the number of inequivalent colouring of X, when the colours available are $c_1, c_2, ..., c_n$ is given by*

$$K_E(c_1, c_2, ..., c_n) = Z_G(\tau_1, \tau_2, ..., \tau_n)$$

where $\tau_1 = c_1^i + c_2^i + , ..., + c_n^i) \forall (1 \le i \le n)$

***Proof:***

We shall begin by finding an alternative formula for

$$K_E(c_1, c_2, ... c_n) = \sum_{\omega \in E} ind\,(\omega)$$

where E is a set of colouring containing one representative of each orbit of $\hat{G}$ on $\Psi$.

Applying this result to the action of $\hat{G}$ on $\Psi$, we get

$$\sum_{\omega \in E} ind\,(\omega) = \frac{1}{|\hat{G}|} \sum_{\hat{g} \in \hat{G}} \left[ \sum_{\omega \in F(\hat{g})} ind\,(\omega) \right]$$

Now the sum in the bracket is just $K_{F(\hat{g})}$, by definition. Further more, a colouring $\omega$ is in $K(\hat{g})$, if and only if it is constant on each cycle of g. Hence the explicit form of $K_{F(\hat{g})}$ is given by

$$K_{F(\hat{g})}(c_1, c_2, \ldots, c_n) = (c_1^{m_1} + , ..., + c_n^{m_1}) , ..., (c_1^{m_k} + , ..., + c_n^{m_k})$$
$$= \tau_{m_1}, ..., \tau_{m_k}$$

Where $m_1, m_2, \ldots, m_k$ are the lengths of the cycles of $g$. In other words, if $g$ has $j_i$ cycles of length $i(1 \leq i \leq n)$ then

$$K_{F(\hat{g})}(c_1, c_2, \ldots, c_n) = (\tau_1^{j1}, \tau_2^{j2}, \ldots, \tau_n^{jn})$$
$$= Z_g(\tau_1, \tau_2, \ldots, \tau_n)$$

Since the representation $g \to \hat{g}$ is a bijection. we have $|G| = |\hat{G}|$, and substituting for $K_{F(\hat{g})}$ above we get.

$$K_E(c_1, c_2, ,\ldots, c_n) = Z_g(\tau_1, \tau_2, \ldots, \tau_n)$$

as required. ■

## 8.9 Burnside's Lemma

Let $G$ be a finite group that acts on the finite set $X$, Let $r$ denote the number of orbits in $X$ under the action of $G$. Then

$$r = \frac{1}{|G|}\sum_{g \in G}|Xg|$$

***Proof:***

Suppose the set $M = \{g, x) \in \underline{G} \times X \,|g.x = x\}$ contains $m$ elements. The idea is to count the elements of $M$ in two different ways, and thereby obtain two different expressions, both equal to $m$. Combining these expressions will yield Burnside's Lemma. Now M contains all pairs $(g, x)$ such that $g.x = x$. We recall that the set $X_g$ for each fixed $g \in G$. contains all $x \in X$ such that $g.x = x$. Thus, for each fixed $g \in G$, there must be $|X_g|$ elements x that fulfills $g.x = x$. Therefore

$$m = \sum_{g \in G}|X_\gamma|$$

The isotropy subgroup $G_x$ contains, on the other hand, those elements $g \in G$ fulfilling $g.x = x$, for each fixed $x \in X$. So for each fixed x, there are $|G_x|$ elements $g$ such that $g.x = x$. This yields

$$m = \sum_{x \in X}|G_x|$$

$\because$ $|orb_G(x)| = |G| / |G_x|$. Thus

$$\sum_{x \in X}|G_x| = \sum_{x \in X}\frac{|G|}{|orb_G(x)|} = |G|\sum_{x \in X}\frac{1}{|orb_G(x)|}$$

Let $B_1, B_2, \ldots, B_r$, denote all orbits in $X$. For each $x \in X$ we have $orb_G(x) = B_i$ for some $i$. Since $X = B_1 \cap B_2 \cup, \ldots, \cup B_r$ and $B_i \cap B_j = \phi$ on $i \neq j$ (the orbits are equivalence classes) we get.

$$\sum_{x \in X}\frac{1}{|orb_G(x)|} = \sum_{x \in B_1}\frac{1}{|B_1|} + \sum_{x \in B_2}\frac{1}{|B_2|} + ,\ldots, + \sum_{x \in B_r}\frac{1}{|B_r|}$$

But for each *i*.

$$\sum_{x \in B_i} \frac{1}{|B_i|} = |B_i| \cdot \frac{1}{|B_i|} = 1$$

and therefore

$$m = |G| \sum_{x \in X} \frac{1}{|orb_G(x)|} = |G|\,(1 + 1 + ,\ldots, + 1) = |G|\, r$$

In the beginning of proof we also found that $m = \sum_{g \in G} |X_g|$. If we combine these expressions for *m*, we obtain an equation which we solve for r, in order to obtain Burnside's Formula.

That is

$$r = \frac{1}{|G|} \sum_{g \in G} |X_g|$$ ■

■ **Example 8.5:** *In how many ways, can we colour the corners of the square with two colours?*

**Solution.**

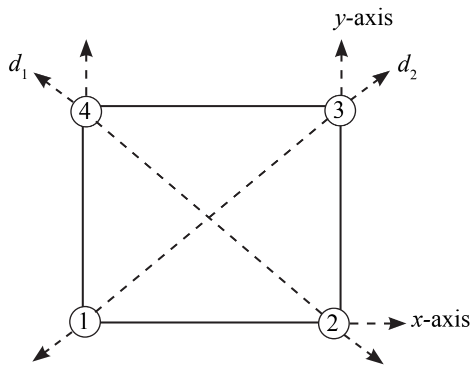



**Fig. 8.5 Square**

In the given figure, the permutation group is $D_4$. We have only two colours.

At initial level, we find all possible permutations under the actions of rotation and reflection, and the sets of all permutations are isomorphic to $D_4$.

i.e., $$\varepsilon = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = (1)\,(2)\,(3)\,(4)$$

Possible ways under actions ε are

$$|X_\varepsilon| = 2^4 = 16 \qquad \ldots (i)$$

We have

$$\rho_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = (1\ 2\ 3\ 4)\ (90° \text{ Anti clock wise})$$

$$\rho_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = (1\ 3)\ (2\ 4)\ (180° \text{ Anti clock wise})$$

$$\rho_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} = (1\ 4\ 3\ 2)\ (90° \text{ clock wise})$$

$\therefore$ Possible ways under action $\rho_1$, $\rho_2$, $\rho_3$ are

$$|X_{\rho_1}| = |X_{\rho_3}| = 2^1 = 2 \qquad \ldots (ii)$$

$$|X_{\rho_2}| = 2^2 = 4$$

By reflections of square along axis, we get.

$$\sigma_x = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} = (1\ \ 4)\ (2\ \ 3)$$

[Rotating along $x$– axis]

$$\sigma_y = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = (\ 1\ 2\ )\ (3\ \ 4)$$

[Rotating along $y$– axis]

$\therefore$ Possible ways under actions $\sigma_x$ *and* $\sigma_y$ *are*

$$|\sigma_x| = |\ \sigma_y| = 2^2 = 4 \qquad \ldots (iii)$$

By reflection of square along diagonals, we get

$$\sigma d_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} = (\ 1\ 3)\ (2)\ (4)$$

[Relation along $d_1$ diagonal]

$$\sigma d_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} = (1)\ (3)\ (2\ 4)$$

[Relation along $d_2$ diagonal]

$\therefore$ Possible ways under action $\sigma d_1$ and $\sigma d_2$ are

$$|\sigma d_1| = |\sigma d_2| = 2^3 = 8 \qquad \ldots (iv)$$

$\therefore$ Required number of colouring $= \dfrac{1}{8}(16\ \ + 2.2 + 4 + 2*4 +\ \ 2*8]$

$$= \frac{1}{8}(16 + 4 + 4 + 8 + 16)$$

$$= \frac{1}{8}\ (24 + 24)\ \ = \frac{1}{8}\ (48) = 6$$

Using Pölya's Enumeration theorem for two colours in a square.

The cycle of permutation will be.

(1) (2) (3) (4)
(1 2 3 4)
(1 3) (2 4)
(1 4 3 2)
(1 4) (2 3)
(1 2) (3 4)
(1 3) (2) (4)
(1) (3) (2 4)

Hence the cycle index of $D_4$ will be

$$Z(D_4) = \frac{1}{8}\ (1.\ a_1^4 + 3a_2{}^2 + 2a_1^2 a_2 + 2a_4) \qquad \text{... } (v)$$

Generating function colouring one corner is

$$F\ (x, y) = 1.\ x + 1.\ y$$

we may use colour $x$ or $y$ to colour a corner in the square.
By Polya's Enumeration Theorem, we have

$$Z(D_4)\ (x, y) = \frac{1}{8}\,[(x + y)^{\,4} + 3\ (x^2 + y^2)^2 + 2\ (x{+}y)^2\ (x^2 + y^2) + 2(x^4 + y^4)]$$

Generating functions for the colouring of the square we have

$Z(D_4)\ (x, y) = x^4 + y^4 + x^3y + xy^3 + 2x^2y^2$

$Z(D_4)$ (1. 1) = 1 + 1 + 1 + 1 + 2 = 6 (Replacing the colour by 1)

Total number of colouring = 6

Hence the possible colouring be:
1 – All corners have colour $x$.
1 – All corners have colour $y$.
1 – Three corners have colour $x$ and one has $y$.
1 – One corner has colour $x$ and three have $y$.
2. Two have colour $x$ and two have $y$. ■

■ **Example 8.6:** *In how many ways can we colour the corners of a square by using three different colour*?

## Solution

In example 8.5 we have 2 colours but here we have 3 colours, so replacing 2 by 3 in the base of number of colouring, we get

$$|X_E| = 3^4 = 81$$
$$|X_{\rho_1}| = |\ X_{\rho_3}| = 3^1 = 3$$
$$|X_{\rho_2}| = 3^2 = 9$$
$$|\sigma_x| = |\sigma_y| = 3^2 = 9$$
$$|\sigma_{d_1}| = |\sigma_{d_2}| = 3^3 = 27$$

$\therefore$ Required number of colourings

$$= \frac{1}{8}\ ( 81 + 2*3 + 9 + 2*9 + 2*27)$$

$$= \frac{1}{8}\ (81 + 6 + 9 + 18 + 54)$$

$$= \frac{1}{8}\ (168) = 21$$

$\therefore$ Cycle index of $D_4$ is.

$$Z(D_4) = \frac{1}{8}\ (1.\ a_1{}^4 + 3a_2^2 + 2a_1^2 a_2 + 2a_4)$$

Generating function colouring one corner is.

$$F(x, y, z) = 1.\,x\ + 1.\,y + 1\ .z$$

This shows that, we can use colour $x$, $y$ or $z$ to colour a corner of the square.

By using Polya's Enumeration theorem, we have

$$Z(D_4)(x, y, z) = \frac{1}{8}\ [(x + y + z)^4 + 3\,(x^2 + y^2 + z^2)^2 + 2\,(x + y + z)^2\,(x^2 + y^2 + z^2)$$
$$+ 2(x^2 + y^4 + z^4)]$$

Generating function for the colouring of the square will be

$$Z(D_4)\ (x, y, z) = x^4 + x^3y + 2x^2y^2 + xy^3 + \text{y} + x^3z + 2x^2yz + 2xy^2z + 2x^2z^2 +$$
$$2xyz^2 + 2y^2z^2 + xz^3 + yz^3 + z^4$$

$$Z(D_4)\,(1, 1, 1) = 1 + 1 + 2 + 1 + 1 + 1 + 2 + 2 + 1 + 1 + 2 + 2 + 2 + 1 + 1$$
$$+ 1 + 1 = 21$$

Hence total number of colouring = 21

$\therefore$ When we use two colours at a time the possible colourings are = 12

■

■ **Example 8.7 :** *In the how many ways can we colour the corners of a square by using four different colours?*

**Solution:**

In examples 8.5 we have 2 colour of colourings, we have replacing 2 by 4 in the base of number of colouring, we have

$$|X_\varepsilon| = 4^4 = 256$$

$$|X_{\rho_1}| = |X_{\rho_1}| = 4^1 = 4$$

$$|X_{\rho_2}| = 4^2 = 16$$

$$|\sigma_x| = |\sigma_y| = 4^2 = 16$$

$$|\sigma_{d_1}| = |\sigma_{d_2}| = 4^3 = 64$$

Required number of colouring = $\frac{1}{8}(256 + 2*4 + 16 + 2*64)$

$$= \frac{1}{8}(440)$$

$$= 55$$

Cycle index of $D_4$ is

$$Z(D_4) = \frac{1}{8}(1.\, a_1^4 + 3a_2^2 + 2a_1^2 \,.\, a_2 + 2a_4)$$

Generating functions colouring one corner is

$$F(X, Y, Z, U) = 1.\, X + 1.\, Y + 1.\, Z + 1.\, U$$

We can use colour $X$, $Y$, Z or $U$ colour corner of the square.

By Polya's Enumeration Theorem, We have

$$Z(D_4)(X, Y, Z, U) = \frac{1}{8}[(X + Y + Z + U)^4 + (X^2 + Y^2 + Z^2 + U^2)^2 + (X + Y + Z + U)^2 + (X^2 + Y^2 + Z^2 + U^2) + 2(X^4 + Y^4 + Z^4 + U^4)]$$

Generating function for the colourings of the square. we have,

$$Z(D_4)(X, Y, Z, U) = U^4 + U^3X + 2\,U^2X^2 + UX^3 + X^4 + U^3Y + 2UX^2Y + X^3Y + 2\,U^2Y^2 + 2\,UXY^2 + 2\,X^2Y^2 + UY^3 + XY^3 + Y^4 + U^3Z + 2U^2XZ + 2UX^2Z + X^3Z + 2U^2YZ + 3\,UXYZ + 2X^2YZ + 2UY^2Z + 2XY^2Z + Y^3Z + 2U^2Z^2 + 2UXZ^2 + 2X^2Z^2 + 2UYZ^2 + 2XYZ^2 + 2Y^2Z^2 + UZ^3 + XZ^3 + YZ^3 + Z^4$$

$$Z(D_4)(1, 1, 1, 1) = 55$$

So total number of colourings = 55

Hence from above we can say that, when we use three colours at a time, the possible colouring are 24. ■

## SUMMARY

1. Expression $\begin{pmatrix} \frac{n(n-1)}{2} \\ e \end{pmatrix}$ can be used to obtain the number of simple labelled graphs of $n$ vertices and $(n-1)$ edges.
2. In a rooted graph one vertex is marked as the root. For each of the $n^{n-2}$ labelled tree we have $n$ rooted labelled trees because any of the $n$ vertices can be mode a root. The number of different rooted labelled tree with $n$ vertices is $n^{n-1}$.

3. To find the number of trees it is necessary to start by counting rooted trees. A rooted tree has one point, its root, distinguished from the others.
4. When a positive integer $P$ is expressed as a sum of positive integers $P = \lambda_1 + \lambda_2 + \lambda_3 + ,\ldots, + \lambda_n$ such that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq ,\ldots, \lambda_n \geq 1$, the $n$-tuple is called a partition of integer $P$.
5. A generating function $f(x)$ is a power series $f(x) = a_0 + a_1x + a_2x^2 + ,\ldots, + a_kx^k$, with some dummy variable $x$. The coefficient $a_k$ of $x^k$ is the desired number, which depends upon a collection of k objects being enumerated.
6. The problem of enumeration of unlabelled trees is more involved and required familiarity with the concepts of generating functions and partitions.
7. Every tree has either one centroid or two centroids. If a tree has two centroids, the centroids are adjacent.
8. On a finite set A of some objects, a permutation $\pi$ is a one-to-one mapping from a onto itself.
9. The number of permutations $m$ in a permutation group is called its order, and the number of elements in the object set on which the permutations are acting is called the degree of the permutation group.
10. Which the $n$ vertices of a group G are subjected to permutation the $\frac{n(n-1)}{2}$ unordered vertex pair also get permuted.

## EXERCISES

1. Prove that the number of different rooted, labelled trees with n vertices is $n^{n-1}$.
2. Prove that the number of labelled graph with $P$ points is $2^{\binom{P}{2}}$.
3. State and prove power Group enumeration theorem.
4. Prove that

$$|\Gamma|\sum_{i=1}^{l}\omega(O_i) = \sum_{\alpha\in\Gamma}\sum_{x\in F(\alpha)}\omega(x).$$

5. If $X = \{1, 2, 3, 4\}$ and $\Gamma = \{1, (1\ 2), (3\ 4)\ (1\ 2)\ (3\ 4)\}$. What is the value of $N(\Gamma)$ ? [Ans. 2}
6. Consider all bracelets made up of 5 beads. The beads can be red, blue and green, and two bracelets are considered to be identical if one can be obtained from the other by rotation. How many distinct bracelets are there? [Ans. 5]
7. In how many essentially ways can we colour the six faces of a cube with atmost three colours? [Ans. 57]

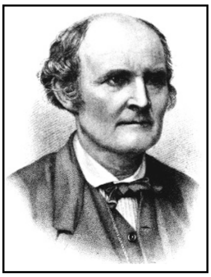
8. Suppose there is a $k$-regular graph $G$ of order $n = k^2 + 1$ and diameter 2. Then prove that $k = 2, 3, 7$ or 57.
9. Prove that there are $n^{n-2}$ vertices on n labelled vertices.
10. State and prove Pölya's Enumeration Theorem.

## Suggested Readings


1. **J.A. Bondy** and **U.S.R. Murty**, *Graph Theory*, Springer, 2008.
2. **R. Diestel**, *Graph Theory*, Springer, 2016.
3. **J. Gross** and **J. Yellen** (Editors), *Handbook of Graph Theory*, CRC Press, 2003.
4. **J. Gross** and **J. Yellen**. *Graph Theory and Its Applications, Second Edition*, Chapman & Hall/CRC, 2006.
5. **D.B. West**, *Introduction to Graph Theory*, Prentice Hall, 2001.


❑❑❑

# 9 Spectral Properties of Graphs



**Arthur Cayley**
**(1821-1895)**

Arthur Cayley was a Britih mathematician who helped to find the modern British school of pure mathematics. He was born on August 16, 1821 in Richmond, Surrey (UK). He studied in Trinity College, Cambridge and excelled Greek, French, German Italian and Mathematics with his new modes as Algebraic Geometry, Cayley_Hamilton Theorem, Group Theory , Graph Theory, Cayley-Hamilton Theorem, Group Theory, Graph Theory, Cayley-Dichson Construction etc. For his outstanding contribution to mathematics he was awarded by Smith Prize (1842), De Morgan Medal (1884). Royal Medal (1859) and Copley Medal (1882). He was a fellow of Royal Society.
He left this planet on January 26, 1895 (Aged 73)

## 9.1 Introduction

The properties of a graphs can be judged by the entire knowledge of their vigenvalues. The set of eigenvalues of a particular graph G is called spectrum of G and it is denoted by Sp(G). For some well known families of graphs like the family of complete graphs, the family of cycles, the spectra can be computed. Some well known mathematicians like Such, Cayley and Ramanujan have some special contribution in spectral graph systems. We can also obtain the spectra of product graphs like cartesian product, direct product and strong product.

Let $G$ be a graph of order $n$ with vertex $V = \{v_1, v_2, \ldots, v_n\}$. The adjacency matrix of G is the n by a matrix A = $\{a_{ij}\}$

where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is adjacent to } v_j \text{ in } G \\ 0 & \text{otherwise} \end{cases}$$


S. K. Yadav, *Advanced Graph Theory*,
https://doi.org/10.1007/978-3-031-22562-8_9

A is a real symmetric matrix of order n.

Hence

(*i*) The spectrum of A (*i.e*. the set of its eigen values) is real.

(*ii*) $\mathbb{R}^n$ has an orthonormal basis of eigenvalues of A.

(*iii*) The sum of the entries of the *i*th row or column of A is $d(v_i)$ in $G$.

The spectrum of $G$ *i.e.* Sp(G) depends on the labelling of vertex set $V$ of $G$.

Let $A'$ be the adjacency matrix of $G$ with respect to this labelling can be obtained from the original labelling by means of a permutation $\pi$ of $V(G)$.

Any such permutation can be effected by means of a permutation matrix $P$ of order $n$.

Let $P = (p_{ij})$, given the new labelling of $V$ i.e. given the permutation $\pi$ on $\{1, 2, .... , n)$ and the vertices $v_i$ and $v_j$, there exists unique $\alpha_0$ and $\beta_0$ such that $\pi(i) = \alpha_0$ and $\pi(j) = \beta_0$.

Equivalently $p_{\alpha_0 i} = 1$ and $p_{\beta_0 j} = 1$ for $\alpha \neq \alpha_0$ and $\beta = \beta_0$ and $p_{\alpha i} = 0 = p_{\beta i}$

Thus $(\alpha_0, \beta_0)$ the entry of the matrix $A' = PAP^{-1} = PAP^T$ ($P^T$ is Transpose of $P$)

$$\sum_{k,l=0}^{n} p_{\alpha_0 k} a_{kl} p_{\beta_0 l} = a_{ij}$$

Hence $v_{\alpha_0} v_{\beta_0} \in E(G)$ iff $v_i v_j \in E(G)$

We arrange the eigenvalues of $G$ in their nondecreasing order

$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq, \ldots, \geq \lambda_n$

If $\lambda_1, \ldots, \lambda_s$ are the distinct eigenvalus of $G$ and if $m_i$ is the multiplicity of $\lambda_i$ as and eigenvalue of $G$

$$Sp(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda s \\ m_1 & m_2 & \ldots & m_s \end{pmatrix}$$

The characteristic Polynomial of $G$ is the characteristic polynomial of the adjacency matrix of $G$ with respect to some labelling of $G$. It is denoted by $\chi(G : \lambda)$

Hence $\quad \chi(G : \lambda) = \det(xI - A) = \det(P(xI - A)P^{-1})$

$$= \det(xI - PAP^{-1})$$

for any permutation matrix of P, and hence $\chi(G : \lambda)$ is also independent of the labelling of $V(G)$

A circulant of order $n$ is a square matrix of order $n$ in which all the rows are obtainable by successive cyclic shifts of one of its rows.

*e.g.* the circulant with first row $(a_1\ a_2\ a_3)$ is the matrix $\begin{pmatrix} a_1 & a_2 & a_3 \\ a_3 & a_1 & a_2 \\ a_2 & a_3 & a_1 \end{pmatrix}$

### Lemma 9.1

Let $A$ he a circulant matrix of order n with first row $(a_1\ a_2, \ldots, a_n)$. Then $Sp(A) = \{a_1 + a_2\omega + ,\ldots, + a_n\omega^{n-1} : \omega = \text{an nth root of unity}\} = \{a_1 + \zeta^{r} + \zeta^{2r} + ,\ldots, + \zeta^{(n-1)r}, 0 \le r \le n-1$ and $\zeta =$ a primitive nth root of unity$\}$

***Proof:***

The characteristic polynomial of $A$ is the determinant $D = \det(xl - A)$.

Hence,

$$D = \begin{vmatrix} x-a_1 & -a_2 & \ldots & -a_n \\ -a_n & x-a_1 & \ldots & -a_{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ -a_2 & -a_3 & \ldots & x-a_1 \end{vmatrix}$$

Let $C_i$ denote the $i$th column of D, $1 \le i \le n$, and $\omega$, an $n$th root of unity. Replace $C_1$ by $C_1 + C_2\omega + ,\ldots, + C_n\omega^{n-1}$. This does not change $D$. Let $\lambda_\omega = a_1 + a_2\omega + ,\ldots, + a_n\omega^{n-1}$. Then the new first column of $D$ is $(x - \lambda_\omega, \omega(x - \lambda_\omega))^T$, and hence $x - \lambda_\omega$ is a factor of $D$. This gives $D = \prod_{\omega:\omega_n=1} (x-\lambda_\omega)$, and $Sp(A) = \{\lambda_\omega : \omega^n = 1\}$.

## 9.2 Spectrum of the Complete Graph $K_n$

For $K_n$, the adjacency matrix $A$ is given by $A = \begin{pmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 1 & \ldots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \ldots & 0 \end{pmatrix}$

$$\lambda_\omega = \omega + \omega^2 + ,\ldots, + \omega^{n-1}$$

$$= \begin{cases} n-1, & \text{if } \omega = 1 \\ -1, & \text{if } \omega \ne 1 \end{cases}$$

Hence, $\quad Sp(K_n) = \begin{pmatrix} n-1 & -1 \\ -1 & n-1 \end{pmatrix}$

## 9.3 Spectrum of the Cycle $C_n$

Label the vertices of $C_n$ as 0, 1, 2 ,…, $n - 1$ in this order. Then $i$ is adjacent to $i \pm 1 \pmod n$. Hence.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}$$

is the circulant with the first row (0 1 0 ... 0 1). Again, $Sp(C_n) = \{\omega^r + \omega^{r(n-1)} : 0 \le r \le n - 1$, where $\omega$ is a primitive $n$th root of unity$\}$.

Taking $\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ we get

$$\lambda_r = \omega^r + \omega^{r(n-1)}$$

$$= \left(\cos\frac{2\pi r}{n} + i\sin\frac{2\pi r}{n}\right) + \left(\cos\frac{2\pi r(n-1)}{n} + i\sin\frac{2\pi r(n-1)}{n}\right)$$

This simplifies to the following:

($i$) If $n$ is odd, $Sp(C_n) = \begin{pmatrix} 2 & 2\cos\frac{2\pi}{n} & \dots & 2\cos\frac{(n-1)\pi}{n} \\ 1 & 2 & \dots & 2 \end{pmatrix}$

($ii$) If $n$ is even, $Sp(C_n) = \begin{pmatrix} 2 & 2\cos\frac{2\pi}{n} & \dots & 2\cos\frac{(n-1)\pi}{n} & -2 \\ 1 & 2 & \dots & 2 & 1 \end{pmatrix}$

## 9.4 Spectra of Regular Graphs Theorem

Let $G$ be a $k$-regular graph of order $n$. Then

($i$) $k$ is an eigenvalue of $G$.

($ii$) If $G$ is connected, every eigenvector corresponding to the eigenvalue $k$ is a multiple of $I$, (the all 1-column vector of length $n$) and the multiplicity of $k$ as an eigenvalue of $G$ is one.

($iii$) For any eigenvalue $\lambda$ of $G$, $|\lambda| \le k$. (Hence $Sp(G) \subset [-k, k]$).

### *Proof:*

($i$) We have $A1 = k1$, and hence $k$ is an eigenvalue of A.

($ii$) Let $x = (x_1, \dots, x_n)^T$ be any eigenvector of A corresponding to the eigenvalue $k$ so that $Ax = kx$. We may suppose that x has a positive entry (otherwise, take $-x$ in place of $x$) and that $x_j$ is the largest positive entry in $x$. Let $v_{i1}$,

$v_{i2},\ldots, v_{ik}$ be the $k$ neighbours of $v_j$ in $G$. Taking the inner product of the $j$th row of $A$ with $x$, we get $x_{i1} + x_{i2} + \quad ,\ldots, + x_{ik} = k_{xj}$. This gives, by the choice of $x_j$, $x_{i1} - x_{i2} + ,\ldots, x_{ik} = x_j$.

Now start at $v_{i1}, v_{i2},\ldots, v_{ik}$ in succession and look at their neighbours in $G$. As before, the entries $x_p$ in $x$ corresponding to these neighbours must all be equal to $x_j$. As $G$ is connected, all the vertices of $G$ are reachable in this way step by step. Hence $x = x_j\,(1, 1,\ldots,\ 1\ )^T$, and every eigenvector $x$ of $A$ corresponding to the eigenvalue $k$ is a multiple of 1. Thus, the space of eigenvectors of $A$ corresponding to the eigenvalue $k$ is one-dimensional, and therefore, the multiplicity of $k$ as an eigenvalue of $G$ is 1.

(*iii*) The proof is similar to (*ii*). If $Ay = \lambda y$, $y \neq 0$, and if $y_j$ is the entry in $y$ with the largest absolute value, we see that the equation $\sum_{p=1}^{k} y_{ip} = \lambda_{yj}$ implies that $|\lambda||y_j| = \left|\sum_{p=1}^{k} y_{ip}\right| \leq \sum_{p=1}^{k} |\, y_{ip}\,| \leq k|y_j|$. Thus, $|\lambda| \leq k$.

**Corollary:** If $\Delta$ denotes the maximum degree of $G$, then for any eigenvalue $\lambda$ of $G$, $|\lambda| \leq \Delta$.

***Proof :***

Considering a vertex $v_j$ of maximum degree $\Delta$, and imitating the proof of (*iii*) above, we get $|\,\lambda||y_j| \leq \Delta|y_j|$.

## 9.5 Theorem of the Spectrum of the Complement of a Regular Graph

Let $G$ be a $k$-regular connected graph of order $n$ with spectrum $\begin{pmatrix} k & \lambda_2 & \lambda_3 & \ldots & \lambda_s \\ 1 & m_2 & m_3 & \ldots & m_s \end{pmatrix}$. Then the spectrum of $G^c$, the complement of $G$, is given by

$$Sp(G^c) = \begin{pmatrix} n-1-k & -\lambda_2-1 & -\lambda_3-1 & \ldots & -\lambda_s-1 \\ 1 & m_2 & m_3 & \ldots & m_s \end{pmatrix}$$

***Proof:***

As $G$ is $k$-regular, $G^c$ is $n - 1 - k$ regular, and hence $n - 1 - k$ is an eigenvalue of $G^c$. Further, the adjacency matrix of $G^c$ is $A^c = J - 1 - A$, where $J$ is the all-1 matrix of order $n$, $l$ is the identity matrix of order $n$, and $A$ is the adjacency matrix of $G$. If $\chi(\lambda)$ is the characteristic polynomial of A. $\chi(\lambda) = (\lambda - k)\chi_1(\lambda)$. By Cayley-Hamilton theorem, $\chi(A) = 0$ and hence we have $A\chi_1(A) = k\chi_1(A)$. Hence, every column vector of $\chi(A)$ is an eigenvector of A corresponding to

the eigenvalue $k$. But the space of eigenvectors of A is generated by 1, $G$ being connected. Hence, each column vector of $\chi_1(A)$ is a multiple of 1. But $\chi_1(A)$ is symmetric and hence $\chi_1(A)$ is a multiple of $J$, say, $\chi_1(A) = \alpha J$, $\alpha \neq 0$. Thus, $J$ and hence $J - I - A$ are polynomials in $A$ (remember : $A_0 = 1$). Let $\lambda \neq k$ be any eigenvalue of $A$ [so that $\chi_1(A) = 0$]m and $Y$ eigenvector of $A$ corresponding to $\lambda$. Then

$$A^C Y = (J - 1 - A)Y$$

$$A^C Y = \left(\frac{\chi_1(A)}{\alpha} - 1 - A\right)Y$$

$$= \left(\frac{\chi_1(A)}{\alpha} - 1 - \lambda\right)Y$$

$$= (-1 - \lambda)Y$$

Thus, $A^c Y = (-1 - \lambda)Y$, and therefore $-1 - \lambda$ is an eigenvalue of $A^c$ corresponding to the eigenvalue $\lambda(\neq k)$ of $A$.

## 9.6 Sachs' Theorem

This theorem determines the spectrum of the line graph of a regular graph $G$ in terms of $Sp(G)$.

Let G be a labelled graph with vertex set

$$V(G) = \{v_1, v_2, \ldots, v_n\}$$

and edges set $E(G) = \{e_1, e_2, \ldots, e_m\}$

with respect to these labellings, the incidence matrix $B = (b_{ij})$ of $G$, which describes the incidence structure of $G$, is defined as the $m$ by $n$ matrix

$$B = (b_{ij}), \qquad \text{where } b_{ij} = \begin{cases} 1 & \text{if } e_i \text{ is incident to } v_j \\ 0 & \text{otherwise} \end{cases}$$

First of all we should go through the following lamma:

### Lemma 9.2

Let $G$ be a graph of order $n$ and size $m$ with $A$ and $B$ as its adjacency and incidence matrices respectively. Let $A_l$, denote the adjacency matrix of the line graph of $G$. Then

(*i*) $BB^T = A_L + 2l_m$

(*ii*) If $G$ is k-regular, $B^T B = A + kl_n$

***Proof:***

Let $A = (a_{ij})$ and $B = (b_{ij})$. We have

$$(i) \qquad (BB^T)_{ij} = \sum_{p=1}^{n} b_{ip} b_{jp}$$

$$= \text{number of vertices } v_p \text{ that are incident to both } e_i \text{ and } e_j$$

$$= \begin{cases} 1 & \text{if } e_i \text{ and } e_j \text{ are adjacent} \\ 0 & \text{if } i \neq j \text{ and } e_i \text{ and } e_j \text{ are nonadjacent} \\ 2 & \text{if } i = j. \end{cases}$$

(*ii*) Proof of (*ij*) is similar.

***Theorem 9.1 (Sachs' theorem).***

Let $G$ be a $k$-regular graph of order $n$. Then $\chi(L(G): \lambda) = (\lambda + 2)^{m-n}\, \chi(G: \lambda + 2 - k)$. where $L(G)$ is the line graph of $G$.

***Proof:***

We consider the two partitioned matrices $U$ and $V$, each of order $n + m$ (where $B$ stands for the incidence matrix of $G$):

$$U = \begin{bmatrix} \lambda I_n & -B^T \\ 0 & I_m \end{bmatrix},\ V = \begin{bmatrix} I_n & B^T \\ B & \lambda I_m \end{bmatrix}$$

We have

$$UV = \begin{bmatrix} \lambda I_n - B^T B & 0 \\ B & \lambda I_m \end{bmatrix}$$

and

$$VU = \begin{bmatrix} \lambda I_n & 0 \\ \lambda B & \lambda I_m - BB^T \end{bmatrix}$$

Now $\quad \det(UV) = \det(VU)$ yields:

$$\lambda^m \det(\lambda I_n - B^T B) = \lambda^n \det(\lambda I_m - BB^T). \qquad \ldots (i)$$

Replacement of $\lambda$ by $\lambda + 2$ in (*i*) yields

$$(\lambda + 2)^{m-n} \det((\lambda + 2)I_n - B^T B) = \det((\lambda + 2)I_m - BB^T). \qquad \ldots (ii)$$

Hence, by Lemma 9.2

$$\begin{aligned} \chi(L(G): \lambda) &= \det(\lambda I_m - A_L) \\ &= \det((\lambda + 2)I_m - (A_L + 2I_m)) \\ &= \det((\lambda + 2)I_m - BB^T) \\ &= (\lambda + 2)^{m-n} \det((\lambda + 2)I_n - B^T B) \text{ by using } (ii) \end{aligned}$$

$$= (\lambda + 2)^{m-n} \det((\lambda + 2)I_n - (A + kI_n))$$

(by Lemma 9.2)

$$= (\lambda + 2)^{m-n} \det((\lambda + 2 - k)I_n - A)$$

$$= (\lambda + 2)^{m-n} \chi(G: \lambda + 2 - k). \qquad \blacksquare$$

## 9.7 Cayley Graphs and Spectrum

Cayley graph are special type of regular graphs which are constructed out of group.

Let $\Gamma$ be a finite group and $S \subset \Gamma$

Such that

(*i*) $e \notin S$ *i.e., e* is the identity to $\Gamma$

(*ii*) If $a \in S$ then $a^{-1} \in S$

(*iii*) $S$ generates $\Gamma$

If we construct a graph $G$ with $V(G) = \Gamma$ and in which $ab \in E(G)$ if and only if $b = as$ for some $s \in S$. Since $as = as'$ in $\Gamma$ implies that $s = s'$, it follows that each vertex $a$ of $G$ is of degree $|S|$; that is, $G$ is a $|S|$-regular graph. Moreover, if $a$ and $b$ are any two vertices of $G$, there exists $c$ in $\Gamma$ such that $ac = b$. But as $S$ generates $\Gamma$, $c = s_1s_2,\ldots, s_p$, where $s_i \in S$, $1 \le i < p$. Hence, $b = as_1s_2,\ldots, s_p$, which implies that $b$ is reachable from $a$ in $G$ by means of the path $a$ $(as_1)$ $(as_1s_2)$ ,…, $\{as_1s_2,\ldots, s_p = b)$. Thus, $G$ is a connected simple graph. [Condition (*i*) implies that G has no loops.] $G$ is known as the *Cayley graph* Cay($\Gamma$: $S$) of the group $\Gamma$ defined by the set $S$.

We then consider a special family of Cayley graphs. Taking $\Gamma = (\mathbb{Z}n, +)$, the additive group of integers modulo $n$. If $S = \{s_1, s_2,\ldots, s_p\}$, then $0 \in S$ and $s_i \in S$ if and only if $n - s_i \in S$. The vertices adjacent to 0 are $s_1, s_2,\ldots, s_p$, while those adjacent to $i$ are $(s_i + i)(\text{mod } n), (s_2 + i)(\text{mod } n),\ldots, (s_p + i)(\text{mod } n)$. Consequently, the adjacency matrix of Cay($\mathbb{Z}^n$ : $S$) is a circulant and, its eigenvalues are

$\{\omega s_1 + \omega s_2 + ,\ldots, + \omega s_p : \omega = \text{ an } n\text{th root of unity}\}$.

We now take S($\subset \mathbb{Z}n$) to be the set $U_n$ of numbers less than $n$ and prime to $n$. Note that $(a, n) = 1$ if and only if $(n - a, n) = 1$. The corresponding Cayley graph Cay($\mathbb{Z}_n$; $U_n$) is denoted by $X_n$ and called the *unitary Cayley graph* mod $n$ (Note that $U_n$ is the set of multiplicative units in $\mathbb{Z}_n$.) Now suppose that $(n_1, n_2) = 1$. What is the Cayley graph $X_{n_1n_2}$ ? Given $x \in \mathbb{Z}_{n_1n_2}$, there exist unique $c \in \mathbb{Z}_{n_1}$ and $d \in \mathbb{Z}_{n_2}$ such that

$$x \equiv a_1(\text{mod } n_1) \text{ and } x \equiv d(\text{mod } n_2) \qquad \ldots (i)$$

Conversely, given $c \in \mathbb{Z}_{n_1}$ and $d \in \mathbb{Z}_{n_2}$ by the Chinese Remainder Theorem there exists a unique $x \in \mathbb{Z}_{n_1n_2}$ satisfying (*i*). Define $f : \mathbb{Z}_{n_1n_2} \to \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$ (the direct product of the groups $\mathbb{Z}_{n_1}$ and $\mathbb{Z}_{n_2}$) by setting $f(x) = (c,d)$. Then $f$ is an additive group isomorphism with $f(U_{n_1n_2}) = U_{n_1} \times U_{n_2}$.

Two vertices $a$ and $b$ are adjacent in $\mathbb{Z}_{n_1 n_2}$ if and only if $a - b \in U_{n_1 n_2}$. Let

$$a \equiv a_1 \pmod{n_1}, \qquad b \equiv b_1 \pmod{n_1}$$
$$a \equiv a_2 \pmod{n_2} \text{ and } \qquad b \equiv b_2 \pmod{n_2}.$$

Then $(a - b, n_1 n_2) = 1$ if and only if $(a - b, n_1) = 1 = (a - b, n_2)$: equivalently, $(a_1 - b_1, n_1) = 1 = (a_2 - b_2, n_2)$ or, in other words, $a_1 b_1 \in E(X_{n_1})$ and $a_2 b_2 \in E(X_{n_2})$.

The adjacency matrix of $X_n$, as noted before, is a circulant, and hence the spectrum $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ of $X_n$ is given by

$$\lambda_r = \sum_{\substack{i \le j \le n \\ (j.n)=1}} \omega^{rj} \text{ where } \omega = e^{\frac{2\pi i}{n}} \qquad \ldots (ii)$$

The sum in (*ii*) is the well-known Ramanujan sum $c(r, n)$.It is known that

$$c(r, n) = \mu(t_r)\frac{\phi(n)}{\phi(t_r)} \text{ where } t_r = \frac{n}{(r,n)},\ 0 \le r \le n-1 \ldots (iii)$$

In relation (*ii*), $\mu$ stands for the Möebius function. Further, as $t_r$ divides $n$, divides $\phi(n)$, and therefore $c(r, n)$ is an integer for each $r$.

## SUMMARY

1. The set of eigenvales of a group $G$ is known as the spectrum of $G$ and denoted by $S_p(G)$.
2. Let $G$ be a group of order $n$ and size $m$ and let $\chi(G : x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + ,\ldots, + a_n$ be the characteristic polynomial of A. Then
   (*i*) $a_1 = 0$
   (*ii*) $a_2 = -m$
   (*iii*) $a_3 = -$ (twice the number of triangles in $G$)
3. The complete graph $K_{10}$ cannot be decomposed into three copies of the Peterson graph.
4. A linear subgraph of a graph $G$ is a subgraph of $G$ whose components are single edges or cycles.
5. Let $A$ be the adjacency matrix of a simple graph $G$. Then
$$\det A = \sum_H (-)^{e(H)} . 2^{c(H)}$$
   where the summation is overall the spanning linear subgraph $H$ of $G$ and $e(H)$ denote respectively the number of even components and the number of cycles in $H$.

## EXERCISES

1. Find the general method to calculate the coefficients of the characteristic polynomial.
2. How can you calculate spectra of regular graphs?
3. Prove that the complete graphs $K_{10}$ cannot be decomposed into three copies of the Petersen graph.
4. State and prove general theorem for spectrum of the complete Bipertite graph $K_{p.q}$.
5. State and prove Harary's Theorem for linear subgraph.
6. Explain spectra of product graphs by suitable graph.
7. What are Cayley's graph, Unitary Cayley Graph and spectra of the Cayley Graph $X_n$.
8. Prove that the eigenvalues of the unitary Cayley graph $X_n = \text{Cay}(Z_n, U_n)$ are all integers $C(r, n)$, $0 < r \leq n - 1$

## Suggested Readings


1. **Balakrishanan R.**, **Ranganathan K.** *A Textbook of Graph Theory*, Spriner (NY), 2012.
2. **Bollobas B.**, *Modern Graph Theory*, Springer, 2001.
3. **Diestel R.** *Graph Theory*, Springer, 2005.
4. **Godsil** and **Royle**, *Algebraic Graph Theory*, Springer, 2010
5. **Harary F.**, *Graph Theory*, Narosa, 2006
6. **Vasudev C.**, *Graph Theory with Application*, New Age, 2006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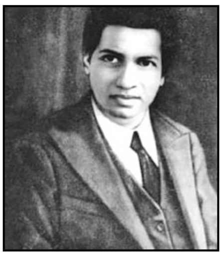
# 10

# Emerging Trends in Graph Theory



S. Ramanujan
(1887–1920)

Srinivasa Ramunujan was a noted Indian mathematician and autodidact who, with almost no formal training in pure mathematics but made extraordinary contribution to mathematical analysis, number theory. He was born on December 22, 1887 in Erode, Madras Presidency in a middle class family. His academic advisors were G.H. Hardy and S.E. Littlewood. He is known for Landau – Ramanujan constant, Mock theta functions, Ramanujan conjecture, Ramanujan prime, Ramanujan – Soldner constant, Ramanujan theta function, Ramanujan's sun, Ramanujan master theorem, Rogers–Ramanujan Graphs etc. He was very much influenced by Prof. G.H. Hardy, a well known English mathematician. During his short life, Ramanujan independently compiled nearly 3900 results. He died on April 26, 1920 (aged 32 years).

## 10.1 Introduction

In this chapter we shall discuss some popular research areas in Graph Theory. In each of the chosen topic, we introduce the problem, give some easily stated theorem, important variation, direction of research and provide reference to survey articles.

## 10.2 Perfect Graphs

A grpah $G$ is said to be perfect iff $\chi(H) = \mathrm{w}(H)$ for every induced subgraph $H$ of $G$. It is common to use stable set to mean an independent set of vertices. As before, clique is a set of pairwise adjacent vertices.

Since we focus on vertex colouring, we restrict our attention to simple graphs. Complementation converts cliques to stable sets and vice-versa, so $w(\overline{H}) = \alpha(H)$.


S. K. Yadav, *Advanced Graph Theory*,
https://doi.org/10.1007/978-3-031-22562-8_10

Properly colouring $\overline{H}$ means expressing $V(H)$ as a union of cliques in $H$; such a set of cliques in $H$ is a clique covering of $H$. For every graph $G$, we have four optimization parameters of interest viz.

(*i*) Independent number $\alpha(G)$ max. size of a stable set.

(*ii*) Clique number $w(G)$ max. size of a clique.

(*iii*) Chromatic number $\chi(G)$ min. size of a colouring.

(*iv*) Clique covering number $\theta(G)$ min. size of a clique covering.

Base defined two types of perfection:

(*a*) $G$ is $\Upsilon$-perfect if $\chi[G(A)] = w[G(A)]$ for all $A \subseteq V(G)$.

(*b*) $G$ is $\alpha$-perfect if $\theta[G(A)] = \alpha[G(A)]$ for all $A \subseteq V(G)$.

The definition of perfect is the same as the definition of $\Upsilon$ perfect. Since $\overline{G}[A]$ is the complement of $G[A]$, the definition of $\alpha$-perfect can be stated in terms of $\overline{G}$ as

$$\text{“}\chi[\overline{G}(A)] = w[\overline{G}(\mathrm{A})]\ \forall A \subseteq V(G)\text{”}.$$

Thus ‘$G$ is $\alpha$-perfect’ has some meaning as ‘$\overline{G}$ is $\Upsilon$-perfect’.

Always $\chi(G) \geq w(G)$ and $\theta(G) \geq \alpha(G)$.

If $k \geq 2$, then $\chi(C_{2k+1}) > w(C_{2k+1})$ and $\chi(\overline{C}_{2k+1}) > w(\overline{C}_{2k+1})$.

Thus, odd cycles and their complements (except $C_3$ and $\overline{C}_3$) are imperfect.

**Duplicating a vertex** $x$ of $G$ produces a new graph $Gox$ by edding a vertex $x'$ with $N(x') = N(x)$. The **vertex multiplication** of $G$ by the nonnegative integer vector $h = (h_1, h_2,\ldots, h_n)$ is the graph $H = Goh$ whose vertex set consists of $\mathrm{h_i}$ copies of each $x_i \in V(G)$ with copies of $x_i$ and $x_j$ adjacent in $H$ iff $x_i \leftrightarrow x_j$ is $G$.

## An Important Lemma

*Vertex multiplication preserves $\Upsilon$-perfection and $\alpha$-perfection.*

### *Proof*:

We observe that $Goh$ can be obtained from an induced subgraph of $G$ by successive vertex duplications. If every $h_i$ is 0 or 1, then $Goh = G[A]$, where $A = [i : h_i > 0)$. Otherwise, start with $G[A]$ and perform duplications until there are $h_i$ copies of $x_i$ (for each $i$ ). Each vertex-duplication preserves the property that copies of $x_i$ and $xj$ are adjacent if and only if $x_i x_j \in E(G)$, so the resulting graph is $Goh$.

If $G$ is $\alpha$-perfect but $Goh$ is not, then some operation in the creation of $Goh$ from $G[A]$ produces a graph that is not $\alpha$-perfect from an $\alpha$-perfect graph. It thus suffices to prove that vertex duplication preserves $\alpha$-perfection. The same reduction holds for $\gamma$-perfection. Since every proper induced subgraph of $Gox$ is an induced subgraph of $G$ or a vertex duplication

of an induced subgraph of $G$, we further reduce our claim to showing that $\alpha(Gox) = \theta(Gox)$ when $G$ is $\alpha$-perfect. When $G$ is $\gamma$-perfect, we extend a proper coloring of $G$ to a proper coloring of $Gox$ by giving $x'$ the same color as $x$. No clique contains both $x$ and $x'$, so $\omega(Gox) = \omega(G)$. Hence $\chi(Gox) = \chi(G) = \omega(G) = \omega(Gox)$.

When $G$ is $\alpha$-perfect, we consider two cases. If $x$ belongs to a maximum stable set in $G$, then adding $x'$ to it yields $\alpha(Gox) = \alpha(G) + 1$. Since $\theta(G) = \alpha(G)$, we can obtain a clique covering of this size by adding $x'$ as a 1-vertex clique to some set of $\theta(G)$ cliques covering $G$.

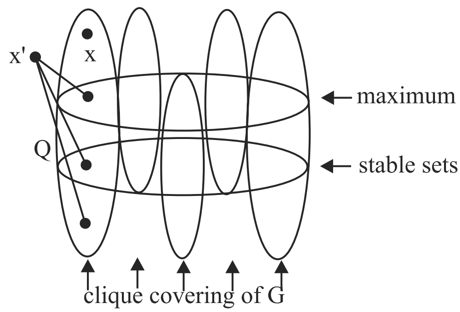



**Fig. 10.1: Clique Coversing**

If $x$ belongs to no maximum stable set in $G$, then $\alpha(Gox) = \alpha(G)$. Let $Q$ be the clique containing $x$ in a minimum clique cover of $G$. Since $\theta(G) = \alpha(G)$, $Q$ intersects every maximum stable set in $G$. Since $x$ belongs to no maximum stable set, $Q' = Q - x$ also intersects every maximum stable set. This yields $\alpha(G - Q') = \alpha(G) - 1$. Applying the $\alpha$-perfection of $G$ to the induced subgraph $G - Q'$ (which contains $x$) yields $\theta(G - Q') = \alpha(G - Q')$. Adding $Q' \cup \{x'\}$ to a set of $\alpha(G) - 1$ cliques covering $G - Q'$ yields a set of $\alpha(G)$ cliques covering $Gox$. ■

### *Theorem 10.1: (Perfect Graph Theorem)*

*A graph is perfect if and only if its complement is perfect.*

### ***Proof***:

It is sufficient to show that $\alpha$-perfection of $G$ implies $\gamma$-perfection of $G$; On applying this to $\overline{G}$ yields the converse. If the claim is faild, then we consider a minimal graph $G$ that is $\alpha$-perfect but not $\gamma$-perfect. By lamma "*In a minimal perfect graph, no stable set intersects every maximum clique*." i.e. If a stable set $S$ in $G$ intersects every $w(G)$ - clique, then perfection of $G - S$ yields $\chi(G - S) = w(G - S) = w(G) - 1$ and $S$ completes a proper $w(G)$-colouring of $G$. This makes $G$ perfect. We may assume that every maximal stable set in $G$ misses some maximum clique $Q(S)$.

We design a special vertex multiplication of $G$. Let $S = (S_i)$ be the list of maximal stable sets of $G$. We weight each vertex by its frequency in $[Q(S_i)]$, letting $h_j$ be the number of stable sets $S_i \in S$ such that $x_j \in Q(S_i)$. $H = Goh$ is $\alpha$-perfect, yielding $\alpha(H) = \theta(H)$. We use counting arguments for $\alpha(H)$ and $\theta(H)$ to obtain a contradiction. (see Fig. 10.2)

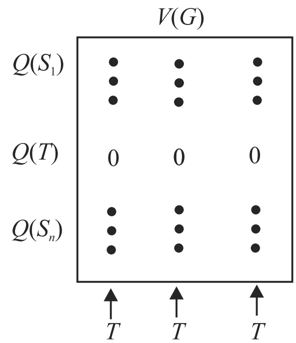



**Fig. 10.2 A perfect Machine**

Let $A$ be the 0, 1-matrix of the incidence relation between $\{Q(S_i)\}$ and $V(G)$; we have $a_{i,j} = 1$ if and only if $x_i \in Q(S_i)$. By construction, $h_j$ is the number of 1s in column $j$ of $A$, and $n(H)$ is the total number of 1s in $A$. Since each row has $\omega(G)$ 1s, also $n(H) = \omega(G)\,|S|$. Since vertex duplication cannot enlarge cliques, we have $\omega(H) = \omega(G)$. Therefore, $\theta(H) \geq n(H)/\omega(H) = |S|$.

We obtain a contradiction by proving that $\alpha(H) < |S|$. Every stable set in $H$ consists of copies of elements in some stable set of $G$, so a maximum stable set in $H$ consists of all copies of all vertices in some maximal stable set of $G$. Hence $\alpha(H) = \max_{T \in s} \sum_{i: x_i \in T} h_i$ *i*. The sum counts the 1s in $A$ that appear in the columns indexed by $T$. If we count these 1s instead by rows, we obtain $\alpha(H) = \max_{T \in s} \sum_{S \in S} |T \cap Q(S)|$. Since $T$ is a stable set, it has at most one vertex in each chosen cligge $Q(S)$. Furthermore, $T$ is disjoint from $Q(T)$. With $|T$ Ç $Q(S)| \leq 1$ for all $S \in$ S, and $|$T Ç Q(T)$| = 0$, we have $\alpha(H) \leq |\mathbf{S}| - 1$. ■

■ **Example 10.1:** *Fractional solutions for an imperfect graph.*

For the 5-cycle, the linear programs for $\omega$, $\chi$, $\alpha$, $\theta$ all have optimal value 5/2. There are five maximal cliques and five maximal stable sets, each of size 2. Setting each $x_j = 1/2$ gives weight 1 to each clique and stable set, thereby satisfying the constraints for either maximization problem. Setting each $y_i = 1/2$ in the dual programs covers each vertex with a total weight of 1, so again the constraints are satisfied. These programs have no optimal solution in integers, and the integer programs have a "duality gap": $\chi = 3 > 2 = \omega$ and $\theta = 3 > 2 = \omega$.

## 10.3 Chordal Graphs Revisited

A chordal graph can be built from a single vertex by iteratively adding a vertex joined to a clique. This is the reverse of a simplicial elimination ordering and we have seen that greedy colouring with respect to such a construction ordering yields an optimal colouring. Many classes of perfect graphs have such a construction procedure that produces the graphs in the class and no others. A construction procedure on the reverse decomposition procedure may lead to fast alogirthms or computations on graphs in the class.

## 10.4 Intersection Representation

An intersection representation of a graph $G$ is a family of sets $\{S_v : v \in V(G)\}$ such that $u \leftrightarrow v$ iff $S_u \cap S_v \neq Q$. If $\{S_v\}$ is an intersection representation of $G$, then $G$ is the intersection graph of $\{S_v\}$.

### *Theorem 10.2: (A Subtree Representation)*

A graph is chordal if and only if it has an intersection representation using subtree of a tree.

### ***Proof***:

First we will have to prove that the condition is equivalent to the existance of a simplicial elimination ordering. We can use inductions with a trivial basis $K_1$.

Let $v_1,\ldots,v_n$ be a simplicial elimination ordering for $G$. Since $v_2,\ldots,v_n$ is a simplicial elimination ordering for $G - v_1$, the induction hypothesis yields a subtree representation of $G - v_1$ in a host tree $T$. Since $v_1$ is simplicial in $G$, the set $S = N_G(v_1)$ induces a clique in $G - v_1$. Therefore, the subtrees of $T$ assigned to vertices of $S$ are pairwise intersecting.

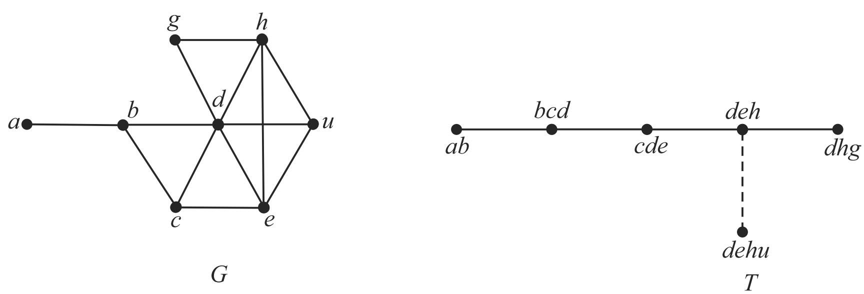



**Fig. 10.3: Subtree Representation**

If $T_1, T_2,\ldots,T_k$ are pairwise intersecting subtrees of a tree $T$ then there is a vertex belonging to all of $T_1, T_2,\ldots, T_k$.

If each vertex $v$ misses some $T(v)$ among $T_1,\ldots,T_k$, we mark the edge that leaves $v$ on the unique path to $T(v)$. If $T$ has $n$ vertices, then

we make $n$ marks, so some edge $uw$ has been market twice. Now $T(u)$ and $T(w)$ have no common vertex. These subtrees have a common vertex $x$. We enlarge $T$ to a tree $T'$ by adding a leaf $y$ a adjacent to $x$, and we add the edge $xy$ to the subtrees representing vertices of $S$. We represent $v_1$ by the subtree consisting only of $y$. This completes a subtree representation of $G$ in $T'$.

Conversely, let $T$ be a smallest host tree for a subtree representation of $G$, with each $v$ Î $V(G)$-represented by $T(v)$ Í $T$. If $xy$ Î $E(T)$, then $G$ must have a vertex $u$ such that $T(u)$ contains $x$ but not $y$; otherwise, contracting $xy$ into $y$ would yield a representation in a smaller tree.

Let $x$ be a leaf of $T$, and let $u$ be a vertex of $G$ such that $T(u)$ contains $x$ but not its neighbor. The subtrees for neighbors of $u$ in $G$ must contain $x$ and hence are pairwise intersecting. Thus $u$ is simplicial in $G$. Deleting $T(u)$ yields a subtree representation of $G – u$. We complete a simplicial elimination order of $G$ using such an ordering of $G – u$ given by the induction hypothesis. ■

■ **Example 10.2:** *The first vertex chosen in the MCS order is arbitrary.*

An application of MCS to the graph G above could start by setting f (c) = 1 and hence 1($b$) = 1($d$) = 1 ($e$) = 1. Next we could select $f$ ($e$) = 2 and update 1($d$) = 2, 1($h$) = 1($u$) = 1. Now d is the only vertex with label as large as 2, and hence $f(d)$ = 3. We update l($b$) = l($h$) = l($u$) = 2, l($g$) = 1, l($a$) = 0. Continuing the procedure can produce the order *c, e, d, b, h, g, a, u* in increasing order of $f$. This is a simplicial construction ordering, and *u, a, g, h, b, d, e, c* is a simplicial elimination ordering.

## 10.5 Tarjan's Theorem (1976)

A simple graphs $G$ is a chordal if and only if the numbering $v_1, v_2,...,v_n$ produced by the maximum cardinality search(MCS) algorithm is a simplicial construction ordering of $G$.

### *Proof*:

To prove the theorem we firstly discuss Maximum Cardinality search (MCS) Algorithm.

***Input:*** *A* graph $G$.

***Output:*** A vertex numbering - a bijection $f : V(G) \to \}1,\ldots, n(G)\}$.

***Idea:*** For each unnumbered vertex $v$, maintain a label $l(v)$ that is its degree among the vertices already numbered. The vertices at the end of a simplicial elimination ordering are those clumped around the last vertex, so in a simplicial construction ordering the vertices with high labels should be added first.

***Initialization:*** Assign label 0 to every vertex. Set $i$ = 1.

***Iteration:*** Select any unnumbered vertex with maximum label. Number. Number it $i$ and add 1 to the label of its neighbors. Augment $i$ and iterate.

If MCS produces a simplicial construction ordering, then $G$ is chordal. Conversely, suppose that $G$ is chordal, and let $f: V(G) \to [n]$ be the numbering produced by MCS. A bridge of $f$ is a chordless path of length at least 2 whose lowest numbers occur at the endpoints. We prove first that $f$ has no bridge. Otherwise, let $P = u, v_1, \ldots, v_k, w$ be a bridge that minimizes mux$\{f(u), f(w)]$. By symmetry, we may assume that $f(u) > f(w)$ ($f$ is used as the vertical eoordinkt,e to position vertices in the illustration).

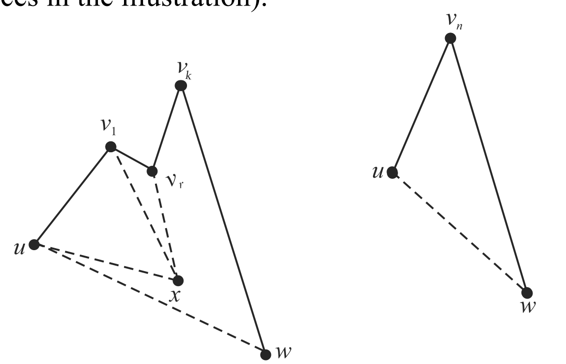



**Fig. 10.4**

Since $u$ is numbered in preference to $v_k$ at time $f(u)$, and $w$ is already numbered at that time, there exists a vertex $x \in N(u) - N(v_k)$ with $f(x) < f(u)$. Letting $v_0 = u$, set $r = \max\{j: x \leftrightarrow v_j\}$. The path $P' = x, v_k, w$ is chordless, since $x \leftrightarrow w$ would complete a chordless cycle. Since both of $f(x), f(w)$ are less than $f(u)$, $P'$ is a bridge that contradicts the choice of $P$. Hence $f$ has no bridge.

With this claim, the proof follows by induction on $n(G)$. It suffices to show that $v_n$ is simplicial, since the application of MCS to $G - v_n$ produces the same numbering $v_1, \ldots, v_{n-1}$ that leaves $v_n$ at the end. If $v_n$ is not simplicial, then $v_n$ has nonadjacent neighbors $u$, $w$, in which case $u, v_n, w$ is a bridge of $f$. ■

***Note*:**

(*i*) *A tree T is a clique tree of G is there is a bijection between V(T) and the maximal cliques of G such that for each, $v \in V(G)$ the cliques containing v induce a subtree of T.*

(*ii*) *Every tree of minimum order in which G has a subtree representation is a clique tree of G.*

(*iii*) *The weighted intersection graph of a collection A of a finite set is a weighted clique in which the elements of A are the vertices and the weight of each edge AA′ is $|A \cap A'|$.*

***Theorem 10.3:*** (McKee [19931])

*Let M(G) be the weighted intersection graph of the set of maximal cliques $(Q_i)$ of a simple graph G. If T is a spanning tree of M(G), then $w(T) \leq \Sigma n(Qi) - n(G)$, with equality if and only T is a clique tree.*

***Proof*:**

Let $T$ be a spanning tree of $M(G)$. Let $T_v$ be the subgraph of $T$ induced by $(Q_i: v \in Q_i\}$. Each vertex $v \in V(G)$ contributes once to the weight of $T$ for each edge of $T_i$; hence $w(T)\ \Sigma_{v \in V(G)} e(T_v)$. Each $T_v$ is a forest, so $e(T_v) \leq n(T_v) - 1$, with equality if and only if $T_v$ is a tree. The term $n(T_v)$ contributes 1 to the size of each clique containing $v$. Summing the inequality for each vertex yields $w(T) \leq \Sigma\, n(Q_i) - n(G)$. Equality holds if and only if each $T_v$ a tree, which is true if and only if $T$ is a clique tree. ■

***Classical Applications of Internal Graphs:***

(*i*) *Analysis of DNA chains:* Interval graphs were invented for the study of DNA. **Benzer** [1959] studied the linearity of the chain for higher organisms. Each gene is encoded as an interval, except that the relevant interval may contain a dozen or more irrelevant junk pieces called "introns" among the relevant pieces called "exons". Under the hypothesis that mutations arise from alterations of connected segments, changes in traits of microorganisms can be studied to determine whether their determining amino-acid sets could intersect. This establishes a graph with traits as vertices and "common alterations" as edges. Under the hypotheses of linearity and contiguity, the graph is an interval graph, and this aids in locating genes along the DNA sequence.

(*ii*) *Timing of traffic lights:* Given traffic streams at an intersection, a traffic engineer (or a person with common sense) can determine which pairs of streams may flow simultaneously. Given an "all-stop" moment in the cycle, the intersection graph of the green-light intervals must be an interval graph whose edges are a subset of the allowable pairs. These can be studied to optimize some criterion such as average waiting time.

(*iii*) *Archeological seriation:* Given pottery samples at an archeological dig, we seek a time-line of what styles. were used when. Assume that each style was used during onetime interval and that two styles appearing in the same grave were used concurrently. Let two styles be an edge if they appear together in a grave. If this graph is an interval graph, then its interval representations are the possible time-lines. Otherwise, the information is incomplete, and the desired interval graph: requires additional edges.

## 10.6 Perfectly Orderable Graph

A perfect order on a graph is a vertex ordering such that greedy colouring with respect to the ordering inherited by each induced subgraph produces an optimal colouring of that subgraph. A perfectly orderable graph is a graph having a perfect order.

In an orientation of $G$, an obstruction is an induced 4-vertex path $a$, $b$, $c$, $d$ whose first and last edges are oriented toward the leaves. The orientation of $G$ associated with a vertex ordering $L$ orients each edge toward the vertex earlier in $L : u \leftarrow v$ if $u < v$. A vertex ordering is obstruction free if its associated orientation has no obstruction.


The orientation associated with a perfect order is obstruction-free, because on an obstruction the greedy coloring would use three colors instead of two. Chvatal proved that a graph is perfectly orderable if and only if it has an obstruction-free ordering. The characterization implies that perfectly orderable graphs are, perfect and that chordal graphs and comparability graphsnre perfectly orderable.

■ ***Example 10.7:*** *Chordal graphs and comparability graphs are perfectly orderable. The orientation cbordal graph associated with a simplicial construction ordering has. no induced $u \leftarrow v \rightarrow w$. A transitive orientation of a comparability graph has no induced $u \rightarrow v \rightarrow w$.*

*Every orientation with an obstruction has both an induced $u \rightarrow v \rightarrow w$ and an induced $u \leftarrow v \rightarrow w$. Hence if G is a comparability graph or a chordal graph, then G has an obstruction-free ordering. By Chvatal's characterization, such graphs are perfectly orderable.*

## 10.7 Minimal Imperfect Graph

A star-cutset of $G$ is a vertex cut $S$ containing a vertex $x$ adjacent to all of $S$ — $\{x\}$. A minimal imperfect graph is an imperfect graph whose proper induced subgraphs are all perfect.

### 10.7.1 *Star-cutset Lemma*

If G has no stable set intersectingeverymaximumclique,andeveryproperinducedsubgraph ofGisw(G)-colourable, then G has no star-cut set.

***Proof***:

Suppose that $G$ has a star-cutset $C$, with $w$ adjacent to all of $C - \{w\}$. Since $G - C$ is disconnected, we can partition $V(G - C)$ into sets $V_1$, $V_2$ with no edge

between them. Let $G_i = G(V_i \cup C)$, and let $f_i$ be a proper $w(G)$-coloring of $G_i$. Let $S_i$ be the set of vertices in $G_i$ with the same color in $f_i$ as $w$; this includes $w$ but no other vertex of $C$. Since there are no edges between $V_1$ and $V_2$, the union $S = S_1 \cup S_2$ is a stable set.

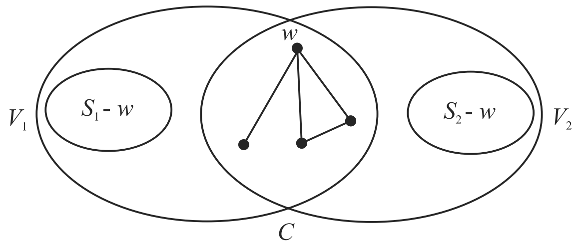



**Fig. 10.5: Star-cutset Graph**

If $Q$ is a clique in $G - S$, then $Q$ is contained in $G_1 - S_1$ or in $G_2 - S_2$. Since $f_i$ provides an $w(G) - 1$-coloring of $G_i - S_i$, we have $|Q| \leq w(G) - 1$. Since this applies to every clique $Q$ in $G - S$, the stable set $S$ meets every $w(G)$-clique of $G$, which contradicts the hypotheses. ■

## 10.8 Imperfect Graphs

*All p-critical graphs are the minimal imperfect graphs.*

Study of p-critical, graphs has benefitted by enlarging the class to include other graphs. Structural properties of the larger class are useful when proving the SPGC for special classes of graphe. **Padberg [1974]** began the study of these graphs. Several definitions were suggested to extend the class of p-critical graphs but turned out to be alternative characterizations of the same class. The definition we use originates in **Bland–Huang–Trotter [1979]**.

For integers $a, w \geq 2$, a graph $G$ is a, $w$-partitionable if it has $aw + 1$ vertices and for each $x \in V(G)$ the subgraph $G - x$ has a partition into a cliques of size $w$ and a partition into $w$ stable sets of size $a$.

The property "$w(G[A])\alpha(G[A]) \geq |A|$ for all $A \subseteq V(G)$" was suggested by Fulkerson; we call it β-perfection. It is implied by α-perfection or γperfection; if we can color $G$ with $w(G)$ stable sets, then some stable set has at least $n(G)/w(G)$ vertices. The converse involves counting arguments like those we gave for the PGT, but more delicate. Since β-perfection is unchanged under complementation.

### *Theorem 10.4:*

*If G is p-critical, then $n(G) = \alpha(G)\omega(G) + 1$. Furthermore, for every $x \in V(G)$, $G - x$ has a partition into $\omega(G)$ stable sets of size $\alpha(G)$ and a partition into $\alpha(G)$ cliques of size $\omega(G)$.*

***Proof*:**

When $G$ is p-critical, the condition for β-perfection fails only for the full vertex set $A = V(G)$. Hence for each $x \in V(G)$ we have

$$n(G) - 1 \le \alpha(G - x)\omega(G - x) = \alpha(G)\omega(G) \le n(G) - 1.$$

Therefore, $n(G) = \alpha(G)\omega(G) + 1$. Since $\chi(G - x) = \omega(G - x) = \omega(G)$, we can cover $G - x$ by $\omega(G)$ stable sets. Having size at most $\alpha(G)$, these sets partition the $\alpha(G)\omega(G)$ vertices of $G - x$ into $\omega(G)$ stable sets of size $\alpha(G)$. Similarly, $\theta(G - x) = \alpha(G - x) = \alpha(G)$ yields a partition of $V(G - x)$ into $\alpha(G)$ cliques of size to $\omega(G)$. ■

■ ***Example 10.4:***

***Cycle-powers:*** The graph $C_n^d$, is constructed by placing $n$ vertices on a circle and making each vertex adjacent to the $d$ nearest vertices in each direction on the circle. When $d = 1$, $C_n^d = C_n$. We view the vertices as the integers modulo $n$, in order.

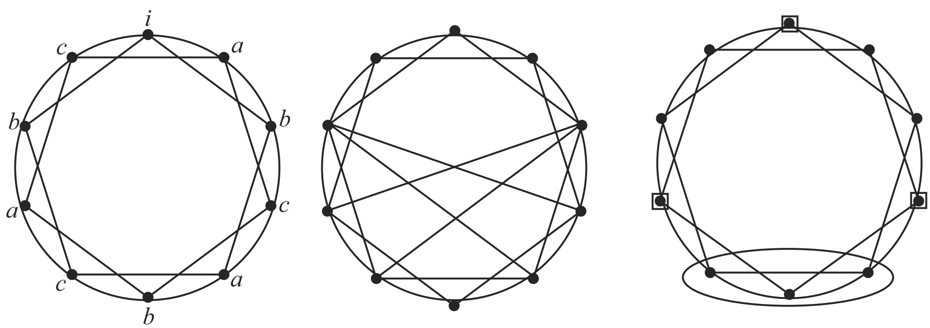



**Fig. 10.6: Cycle Power**

The graph $C_{10}^2$, shown on the left below, is neither perfect nor p-critical (the vertices 0, 2, 4, 6,8 induce $C_5$), but $C_{10}^2$ is 3,3-partitionable. When $i$ is removed, the unique partition of the remaining nine vertices into three triangles is $\{(i + 1, i + 2, i + 3), (i + 4, i + 5, i + 6), (i + 7, i + 8, i + 9)\}$, and the unique partition into three stable sets is $\{(i + 1, i + 4, i + 7), (i + 2, i + 5, i + 8), (i + 3, i + 6, i + 9)\}$.

Always $C_{aw+1}^{w-1}$ is $a$, $w$-partitionable. Every $w$ consecutive vertices in $G - x$ form a clique, and every a vertices spaced by jumps of length $w$ form a stable set. Showing that $C_{aw+1}^{w-1}$ is p-critical if and only if $w = 2$ or $a = 2$ reduces the SPGC to the statement that $G$ is p-critical if and only if $G = C_{a(G)w(G)+1}^{w(G)-1}$.

***Theorem 10.5:***

*A graph G of order n = aw + 1 is a, w-partitionable if and only if both conditions below hold:*

1. *$\alpha(G) = a$ and $\omega(G) = \omega$, and each vertex of G belongs to exactly w cliques of size w and a stable sets of size a.*
2. *G has exactly n maximum cliques $\{Q_i\}$ and exactly n maximum stable sets $(S_j)$, with $Q_i$ Ç $S_j = q$ if and only if $i = j$ ($Q_i$ and $S_i$ are mates).*

***Proof*:**

We have proved $\chi(G - x) = w = \omega(G)$ and $\theta(G - x) = a = \alpha(G)$ for each $x \in V(G)$. Choose a clique $Q$ of size $w$. For each $x \in Q$, $G - x$ has a partition into $a$ cliques of size $w$. Together, $Q$ and these $w$ partitions form a list of $n = aw + 1$ maximum cliques $Q_1,\ldots, Q_n$. Each vertex outside $Q$ appears in one clique in each partition. Each vertex in $Q$ appears in $Q$ and once in $w - 1$ partitions. Hence every vertex appears in exactly $w$ cliques in the list.

For each $Q_i$, we obtain a maximum stable set $S_i$ disjoint from $Q_i$. Choose $x \in Q_i$. The $w$ maximum stable sets that partition $V(G - x)$ can meet $Q_i$, only at the $w - 1$ vertices other than $x$. Therefore, one of these stable sets misses $Q_i$; call it $S_i$. We will show that these two lists contain all the cliques and stable sets and have the desired intersection properties.

Let $A$ be the incidence matrix with $a_{i,j} = 1$ if $x_j \in Q_i$ and $a_{i,j} = 0$ otherwise. Let $B$ be the matrix with $b_{i,j} = 1$ if $x_j \in S_i$ and $b_{i,j} = 0$ otherwise. The $ij$th entry of $AB^T$ is the dot product of row $i$ of $A$ with row $j$ of $B$, which equals $|Q_i \cap S_j|$. By proving that $AB^T = J - I$, where $J$ is the matrix of all 1s, we obtain $Q_i \cap S_j \neq \theta$ if and only if $i \neq j$. Since $J$ - $I$ is nonsingular, this will also imply that $A$ and $B$ are nonsingular. Nonsingular matrices have distinct rows, and hence $Q_1,\ldots$ , $Q_n$ and $S_1,\ldots S_n$ will be distinct.

By construction, $|Q_i \cap S_i| = 0$. Since cliques and stable sets intersect at most once, to prove that $AB^T = J - I$ we need only show that each column of $AB^T$ sums to $n - 1$. Multiplying by the row vector $1_n^T$ on the left computes these sums. We constructed $A$ so that each column has $w$ is (because each vertex appears in $w$ cliques in the list) and $B$ so that each row has $a$ 1s (because each stable set has size $a$). Therefore,

$$1_n^T(AB^T) = (1_n^T A)B^T = w1_n^T B^T = wa1_n = (n - 1)1_n^T \qquad \ldots(i)$$

To prove that $G$ has no other maximum cliques, we let $q$ be the incidence vector of a maximum clique $Q$ and show that $q$ must be a row of $A$. Since $A$ is nonsingular, its rows span $\mathbb{R}^n$, and we can write $q$ as a linear combination: $q = tA$. To solve for $t$, we need $A^{-1}$. Since every row of $A$ sums to $\omega$, we have $A(\omega^{-1} J - B^T) = \omega^{-1}\omega J - (J - 1) = I$, and hence $A^{-1} = \omega^{-1} J - B^T$ . Thus,

$$t = q^{A-1} = q(\omega^{-1} J - B^T) = \omega^{-1}qJ - qB^T = \omega^{-1}\omega 1_n^T - qB^T \qquad \ldots(ii)$$

The $i$th column of $B^T$ is the incidence vector of $S_i$; hence coordinate $i$ of $qB^T$ equals $|Q \cap S_i|$, which is 0 or 1. Hence $t$ is a 0, 1-vector and $q$ is a sum of

rows of $A$. Since $q$ sums to $\omega$, only one row can be used. Thus $q$ is a row of $A$ and $Q_1,\ldots, Q_n$ are the only maximum cliques.

The same argument applied to $\overline{G}$ shows that $G$ has exactly $n$ maximum stable sets, with each vertex appearing in $a$ of them.

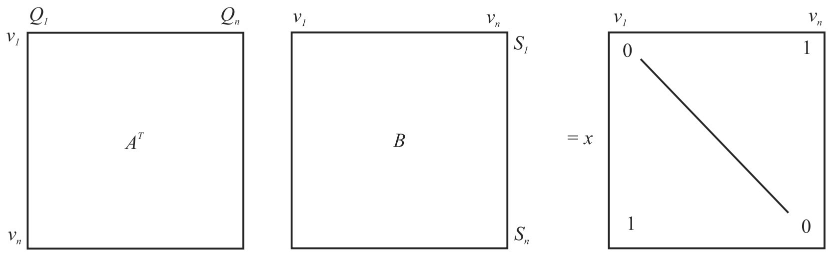



**Fig. 10.7**

Sufficiency. we need only prove that $\chi(G - x) \le w$ and $\theta(G - x) \le a$ for all $x \in V(G)$. Given the cliques and stable sets as guaranteed by condition (2), define the incidence matrices $A$, $B$ as above. By condition (1), each column of $B$ has $a$ 1s, and hence $JB = aJ = BJ$. The intersection requirements in condition (2) yield $AB^T = J - I$. This is nonsingular, so $B$ is nonsingular and

$$A^TB = B^{-1}BA^TB = B^{-1}(J - I)B = B^{-1} BJ - I = J - I.$$

In the product $A^TB = J - I$, the row corresponding to $x \in V(G)$ states that $V(G - x)$ is covered by, the mates of the $w$ maximum cliques containing $x$ (illustrated below), and hence $\chi(G - x) \le w$. Similarly, the column corresponding to $x$ states that $V(G - x)$ is covered by the mates of the $a$ maximum stable sets containing $x$, and hence $\theta(G - x) \le a$. ■

***Note*:**

*An edge of a graph is critical if deleting it increases the independent number. A pair of non-adjacent vertices is co-critical of adding an edge joining them increases the clique number.*

***Theorem 10.6:***

*For an edge xy in a partitionable graph G, the following statements are equivalent.*

(*a*) *xy is a critical edge.*

(*b*) $S \cup \{x\} \in X(G - y)$.

(*c*) *xy belongs to w – 1 maximum cliques.*

***Proof*:**

B $\Rightarrow$ A. $S \cup \{x, y\}$ is a stable set of size $\alpha + 1$ in $G - xy$.

A $\Rightarrow$ C. If $xy$ is critical, then there is a set $S$ such that $S \cup \{x\}$ and

$S \cup \{y\}$ are maximum stable sets in $G$. Hence every maximum clique containing $x$ but not $y$ is disjoint from $S \cup \{y\}$. Since there are $\omega$ maximum cliques containing $x$ and only one maximum clique disjoint from $S \cup \{y\}$, the remaining $\omega - 1$ maximum cliques containing $x$ must also contain $y$.

C $\Rightarrow$ B. The stable sets in the unique coloring of $G - x$ are the mates of the cliques containing $x$. Since $xy$ belongs to $\omega - 1$ maximum cliques, the mates of these $\omega - 1$ cliques belong to both $X(G - x)$ and $X(G - y)$. This leaves only $a + 1$ vertices in the graph, consisting of the vertices $x$, $y$ and a stable set S such that $S \cup \{y\} \in X(G - x)$ and $S \cup \{x\} \in X(G - y)$. ■

## 10.9 Strong Perfect Graph Coryecture

We have been proving properties of partitionable graphs in a "top down" approach to the SPGC, trying to find enough properties to eliminate all but odd cycles and their complements as p-critical graphs. The "bottom up" approach is to verify that the SPEC holds on larger and larger classes of graphs, until all are included.

One way to prove that a class **G** satisfies the SPGC is to prove that every Berge graph in **G** is perfect. A hereditary class **G** satisfies the SPGC if the odd cycles and their complements are the only p-critical graphs in **G**. Every cycle is both a circle graph and a circular-arc graph, but neither of these classes contains the other. One way to prove the SPGC for a class **G** is to show that every partitionable graph in **G** belongs to another class **H** where the SPGC is known to hold. In this role we use the class $\{C_n^d\}$.

***Theorem 10.7:***

*The SPGC holds for circular-arc graphs.*

***Proof*:**

Recall that $N[v]$ denotes $N(v) \cup \{v\}$, the closed neighborhood of $v$. When $G$ is partatisnable with distinct vertices $x$, $y$, we claim that $N[x] \nsubseteq N[y]$. Consider the clique $Q$ containing $x$ in $O(G - y)$; we have $Q \subseteq N[x]$. If $N(y)$ contains $N[x]$, then $Q \cup \{y\}$ is a clique of size $\omega(G) + 1$.

Now, if $G$ is a partitionable circular-arc graph, it suffices to show that $G = C_n^{\omega(G)-1}$, because the SPGC holds for cycle-powers Consider a circular-arc representation that assigns arc $A_x$ to $x \in V$. Since $N[y]$ cannot contain $N[x]$, the arc $A_x$ cannot lie within another arc $A_y$ of the representation. If no arc contains another, then every arc that intersects $A_x$ contains at least one endpoint of $A_x$. Since the vertices corresponding to the arcs containing one point induce a clique, there are at most $\omega - 1$ other arcs containing each endpoint of $A_x$. Equality holds (and no other arc contains both endpoints of $A_x$), since $\delta(G) \geq 2\omega - 2$.

Starting from a given point p on the circle, let $v_i$ be the vertex represented by the $i$th arc encountered moving clockwise from $p$. Since each arc meets exactly $\omega - 1$ others at each endpoint, $v_i$ is adjacent to $v_{i+1}$ (addition modulo $n$) for each $i$. Hence $G = C_n^{w-1}$. ■

## 10.10 Hereditary Family

A hereducta family is a collection of sets, $F$, such that everysubset of a set in $F$ is also in $F$. A hereditary system $M$ on $E$ consists of a nonempty ideal $I_M$ of subsets of $E$ and the various ways of specifying the ideal, called aspects of $M$.

The elements of $I_M$ are the independent sets of $M$. The other subsets of $E$ are dependent. The bases are the maximal independent sets, and the circuts are the minimal dependent sets; $B_M$ and $C_M$ denote these families of subsets of $E$. The rank of a subset of $E$ is the maximum size of an independent set in it. The rank function $r_M$ is defined by $r(X) = \max \{|Y| : Y \subseteq X, Y \in I\}$.

Hereditary systems are too general to behave nicely. We restrict our attention to hereditary systems having an additional property, and these we call matroids. We can translate any restriction on $I_M$ into a corresponding restriction on some other aspect of the hereditary system. Because hereditary systems can be specified in many ways, we have many equivalent definitions of matroids. Using various motivating examples, we state several of these properties that characterize matroids. Later we prove that they are equivalent. We begin with the fundamental example from graphs.

Aspects of hereditary systems. A hereditary system $M$ is determined by. any of $I_M$, $B_M$, $C_{M,}$, $r_M$, etc., because each aspect specifies the others. We have expressed $B_M$, $C_M$, $r_M$ in terms of $I_M$. Conversely, if we know $B_M$, then $I_M$, consists of the sets contained in members of $B_M$. If we know $C_M$, then $I_M$ consists of the sets containing no member of $C_M$. If we know $r_M$, then IM = $\{X \subseteq E : r_M(X) = |X|\}$.

## 10.11 Matroids

Many results of graph theory extend or simplify in the theory of matroids. These include the greedy algorithm for minimum spanning trees, the strong duality between maximum matching and minimum vertex cover in bipartite graphs, and the geometric duality relating planar graphs and their duals.

Matroids arise in many contexts but are special enough to have rich combinatorial structure. When a result from graph theory generalizes to matroids, it can then be interpreted in other special cases. Several difficult theorems about graphs have found easier proofs using matroids.

Matroids were introduced by **Whitney** [1935] to study planarity and algebraic aspects of graphs, by **MacLane** [1936] to study geometric lattices, and by van der **Waerden** [1937] to study independence in vector spaces. Most of the language comes from these contexts. Here we emphasize applications to graphs.

The cycle Matroid $M(G)$ of a graph $G$ is the heredctary system on $E(G)$ whose circuits are the cycles of $G$. A hereditary system that is $M(G)$ for some graph $G$ is called a graphic matroid.

In a hereditary system, a loop is an element forming a circuit of size 1. Parallel elements are distinct non-loops forming a circuit of size 2. A hereditary system is simple if it has no loops or parallel elements.

The vectorial matroid on a set $E$ of vectors in a vector space is the hereditary system whose independent sets are the linearly independent subsets of vectors in $E$. A matroid expressible in this way is a linear ntatroid (or representable matroid). The column matroid $M(A)$ of a matrix $A$ is the vectorial matroid defined on its columns.

The transversal matroid induced by sets $A_1,\ldots, A_m$, with union $E$ is the hereditary system on $E$ whose independent sets are the systems of distinct representatives of subsets of $(A_1,\ldots, A_m)$. Equivalently, letting $G$ be the $E$, $[m]$-bigraph defined by $e \leftrightarrow i$ if and only if $e \in A_i$, the independent sets are the subsets of $E$ that are saturated by matchings in $G$.

### 10.11.1 Hereditary Systems

Label each vertex $a = (a_1,\ldots,a_n)$ of the hypercube $Q_n$ by the corresponding set $X_a = (i : a_i = 1)$. Draw $Q_n$ in the plane so that the vertical coordinates of vertices are in order by the size of the sets labeling them.

The diagram below illustrates the relationships among the independent sets, bases, circuits, and dependent sets of a hereditary system. The bases are the maximal elements of the family **I** and the circuits are the minimal elements not in **I**. In every hereditary system,$\varnothing$ belongs to **I**. If every set is independent, then there is no circuit, but there is always at least one base.

In the example on the right, the independent sets are the acyclic edge sets in a graph with three edges. The only dependent sets are $\{1, 2\}$ and $\{1, 2, 3\}$, the only circuit is $\{1, 2\}$, and the bases are $\{1, 3\}$ and $\{2, 3\}$. The rank of an independent set is its size. For the dependent sets, we have $r(\{1, 2\}) = 1$ and $r(\{1, 2,3\}) = 2$.

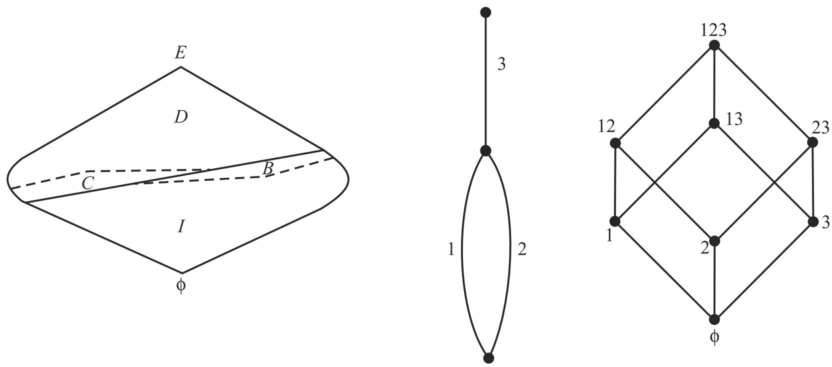



**Fig. 10.8: Graph of Hereditary System**

Bases in cycle matroids. The bases of the cycle matroid $M(G)$ are the edge sets of the maximal forests in $G$. Each maximal forest contains a spanning tree from each component, so they have the same size. Consider $B_1, B_2 \in \mathbf{B}$ with $e \in B_1 - B_2$. Deleting $e$ from $B_1$ disconnects some component of $B_1$; since $B_2$ contains a tree spanning that component of $G$, some edge $f \in B_2 - B_1$ can be added to $B_1 - e$ to reconnect it.

For a hereditary system $M$, the base exchange property is: if $B_1, B_2 \in \mathbf{B}_M$, then for all $e \in B_1 - B_2$ there exists $f \in B_2 - B_1$ such that $B_1 - e + f \in \mathbf{B}_M$. Matroids are the hereditary systems satisfying the base exchange property.

## 10.11.2 Rank Function in Cycle Matroids

Let $G$ be a graph with $n$ vertices. For $X \subseteq E(G)$, let $G_X$ denote the spanning subgraph of $G$ with edge set $X$. In $M(G)$, an independent subset of $X$ is the edge set of a forest in $G_X$. When $G_X$ has $k$ components, the maximum size of such a forest is $n - k$. Hence $r(X) = n - k$. Below we show such a forest $Y$ (bold) within $X$ (bold and solid).

If $r(X + e) = r(X)$ for some $e \in E - X$, then the endpoints of $e$ lie in a single component of $Gx$; adding $e$ does not combine components. If we add two such edges, then again we do not combine components. Therefore, $r(X) = r(X + e) = r(X + f)$ implies $r(X) = r(X + e + f)$.

For a hereditary system $M$ on $E$, the (weak) absorption property is: if $X \subseteq E$ and $e, f \in E$, then $r(X) = r(X + e) = r(X + f)$ implies $r(X + e + f) = r(X)$. Matroids are the hereditary systems satisfying the *absorption property*.

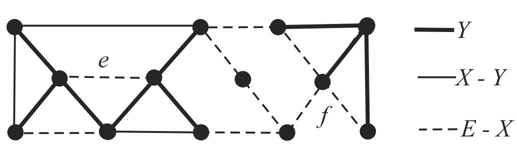



**Fig. 10.9: Cycle Matroids**

Independent sets in transversal matroids. When $M$, $M'$ are matchings in $G$ and $|M'| > |M|$, the symmetric difference $M \Delta M'$ contains an $M$-augmenting path $P$. Replacing $M \cap P$ with $M' \cap P$ yields a matching of size $|M| + 1$ that saturates all vertices of $M$ plus the endpoints of $P$.

Consider independent sets $I_1$, $I_2$ in the transversal matroid generated by $A_1, \ldots, A_m$. In the associated bipartite graph, let $M_1$, $M_2$ be matchings saturating $I_1$, $I_2$, respectively (on the left below, $M_1$, is solid and $M_2$ is dashed). If $|I_2| > |I_1|$ then the snatching obtained from $M_1$ by using $M_1$-*augmenting path* in $M_2 \Delta M_1$ saturates $I_1$, plus an element $e \in I_2 - I_1$; this "augments" $I_2$. For a hereditary system on $E$, the augmentation property is for distinct $I_1, I_2 \in \mathbf{I}$ with $|I_2| > |I_1|$, there exists $e \in I_2 - I_1$ such that $I_1 \cup \{e\} \in \mathbf{I}$. Matroids are the hereiliitary systems satisfying the *augmentation property*.

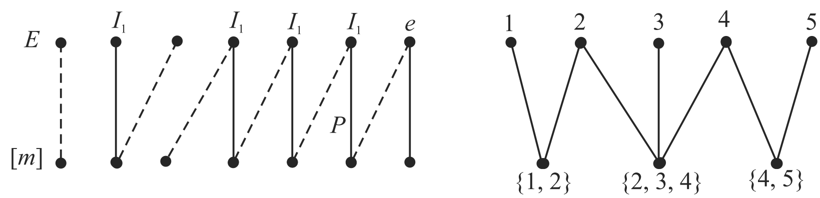



**Fig. 8.10: Transversal Matroid**

The transversal matroid of the family $\mathbf{A} = \{1, 2\}, \{2, 3, 4\}, \{4, 5\}\}$, illustrated by the bipartite graph on the right, is again $M(K_4 - e)$. ■

## 10.12 Basic Properties of Matroids

A hereditary system $M$ on $E$ is a matroid if it satisfies any of the following additional properties, where $I$, $B$, $C$, and $r$ are the independent sets, bases, circuits, and rank function of $M$.

**I: augmentation**—if $I_1, I_2 \in \mathbf{I}$ with $|I_2| > |I_1|$, then $I_1 + e \in \mathbf{I}$ for some $e \in I_2 - I_1$.

**U: uniformity**—for every $X \subseteq E$, the maximal subsets of $X$ belonging to $\mathbf{I}$ have the same size.

**B: base exchange**—if $B_1, B_2 \in \mathbf{B}$, then for all $e \in B_1 - B_2$ there exists $f \in B_2 - B_1$ such that $B_1 - e + f \in \mathbf{B}$.

**R: submodularity**—$r(X \cap Y) + r(X \cup Y) \leq r(X) + r(Y)$ whenever $X, Y \subseteq E$.

**A: weak absorption**—$r(X) = r(X + e) = r(X + f)$ implies $r(X + e + f) = r(X)$ whenever $X \subseteq E$ and $e, f \in E$,

**A′: strong absorption**–if $X, Y \subseteq E$, and $r(X + e) = r(X)$ for all $e \in Y$, then $r(X \cup Y) = r(X)$.

**C: weak elimination**—for distinct circuits $C_1$, $C_2 \in$ **C** and $x \in C_1 \cap C_2$, there is another member of **C** contained in $(C_1 \cup C_2) - x$.

**J: induced circuits**—if $I \in$ **I**, then $I + e$ contains at most one circuit.

**G: greedy algorithm**—for each nonnegative weight function on $E$, the greedy algorithm selects an independent set of maximum total weight.

The *base exchange property* implies that all bases have the same size: if $|B_1| < |B_2|$ for some $B_1, B_2 \in$ **B**, then we can iteratively replace elements of $B_1 - B_2$ by elements of $B_2 - B_1$ to obtain a base of size $|B_1|$ contained in $B_2$, but no base is contained in another.

## 10.13 Span Function

The *span function* of a hereditary system $M$ is the function $\sigma_M$ on the subsets of $E$ defined by $\sigma_M(X) = X \cup \{e \in E: Y + e \in \mathbf{C}_M \text{ for some } Y \subseteq X\}$. If $e \in \sigma(X)$, then $X$ Spans $e$.

In a hereditary system, $X$ is a dependent set if and only if it contains a circuit, holds if and only if $e \in \sigma(X - e)$ for some $e \in X$. We can therefore find the independent sets from the span function via $\mathbf{I} = \{X \subseteq E: (e \in X) \Rightarrow (e \notin \sigma(X - e))\}$.

The properties of span functions that we use in studying matroids are ($s_1$, $s_2$, $s_3$) below (an additional technical condition is needed to characterize the span functions of hereditary systems). First we illustrate property ($s_3$) using graphs.

### *Theorem 10.8:*

*If M is a hereditary system, then each condition below is necessary and sufficient for M to be a matroid:*

**P: incorporation**—$r(s(X)) = r(X)$ for all $X \subseteq E$.

**S: idempotence**—$\sigma^2(X) = \sigma(X)$ for all $X \subseteq E$.

**T: transitivity of dependence**—if $e \in \sigma(X)$ and $X \subseteq \sigma(Y)$, then $e \in \sigma(Y)$.

**C′: strong elimination**—whenever $C_1, C_2 \in C$, $e \in C_1 \cap C_2$, and $f \in C_1 \Delta C_2$, there exists $C \in C$ such that $f \in C \subseteq (C_1 \cup C_2) - e$.

### *Proof*:

U $\Rightarrow$ P: Every element in $\sigma(X) - X$ completes a circuit with a subset of $X$ and thus lies in the span of every set between $X$ and $\sigma(X)$. Thus it suffices to prove that $r(Y + e) = r(Y)$ when $e \in \mathrm{r}(Y)$. Let $Z$ be a subset of $Y$ such that $Z + e \in C$. Augment $Z$ to a maximal independent subset $I$ of $Y + e$. By the uniformity property, $|I| = r(Y + e)$. Since $Z + e \in C$, we have $e \in I$. Thus $I \subseteq Y$, and we have $r(Y) \geq |I| = r(Y + e)$ (Absorption can be used instead.)

P $\Rightarrow$ S: Since $\sigma$ is expensive, $\sigma^2(X) \supseteq \sigma(X)$, and we need only show that $e \in \sigma^2(X)$ implies $e \in \sigma(X)$. By the incorporation property, $r(\sigma(X) + e) = r(\sigma(X))$ and $r(\sigma(X)) = r(X)$. Since $X \subseteq \sigma(X)$, monotonicity of $r$ yields $r(X) \leq r(X + e) \leq r(\sigma(X) + e) = r(X)$. Since equality holds throughout, yields $e \in \sigma(X)$.

S $\Rightarrow$ T: If $X \subseteq \sigma(Y)$, then the order-preserving and idempotence properties of a imply $\sigma(X) \subseteq \sigma^2(y) = \sigma(Y)$.

T $\Rightarrow$ C′: Given distinct $C_1, C_2 \in C$ with $e \in C_1 \cap C_2$ and $f \in C_1 - C_2$, we want $f \in \sigma(Y)$, where $Y = (C_1 \cup C_2) - e - f$. We have $f \in \sigma(X)$, where $X = C_1 - f$. By T, it suffices to show $X \subseteq \sigma(Y)$. Since $X - e \subseteq Y \subseteq \sigma(Y)$, we need only show $e \,\varepsilon\, \sigma(Y)$. Since $\sigma$ is order-preserving, we have $e \in \sigma(C_2 - e) \subseteq \sigma(Y)$.

C′ $\Rightarrow$ C: $C$ is a leas restrictive statement than C′. ■

***Note***:

(*i*) *The dual of a hereditary system M on E is the hereditary system M* whose bases are the complements of the bases of M. The aspects* $B^*(B_{M^*})$, $C^*$, $I^*$, $r^*$, $\sigma^*$, *of M* are the cobases, cocircuits, etc., of M.*

*The supbases **S** of M are the sets containing a base. The hypobases **H** are the maximal subsets containing nobase. We write* $\overline{X}$ *for* $E - X$.

(*ii*) *The bond malroid or cocycle matroid of a graph G is the heredctary system whose circuits are the bonds of G.*

(*iii*) *For a hereditary system M on E, the restriction of M to* $F \subseteq E$, *denoted* $M \mid F$ *and obtained by deleting* $\overline{F}$, *is the hereditary system defined by* $I_{M \mid F} = (X \subseteq F: X \in I_M)$. *The contraction of M to* $F \subseteq E$, *denoted M.F and obtained by contracting* $\overline{F}$, *is the hereditary system defined by* $S_{M \cdot F} = (X \subseteq F: X \cup \overline{F} \in S_M)$. *When* $F = E - e$, *we write* $M - e = M \mid F$ *and* $M . e = M. F$. *The minors of M are the hereditary systems arising from M using deletions and contractions.*

### *Theorem 10.9: Matroid Intersection Theorem (1970)*

*For matroids* $M_1$, $M_2$ *on E, the size of a largest common independent set satisfies* $max\ (\mid I \mid : I \in \boldsymbol{I}_1 \cap \boldsymbol{I}_2) = min_{X \subseteq E}(r_1(X) + r_2(\overline{X}))$.

***Proof***:

For weak duality, consider arbitrary $I \in I_1 \cap I_2$ and $X \subseteq E$. The sets $I \cap X$ and I Ç $\overline{X}$ are also common independent sets, and $\mid I \mid = |I \cap X| + |I \cap \overline{X}| \leq r_1(X) \mid r_2(\overline{X})$.

To achieve equality, we use induction on $|E|$; when $|E| = 0$ both sides are 0. If every element of $E$ is a loop in $M_1$ or in $M_2$, then max $|I| = 0 = r_1(X) + r_2(\overline{X})$, where $X$ consists of all loops in $M_1$. Hence we may assume that $|E| > 0$ and that some $e \in E$ is a non-loop in both matroids. Let $F = E - e$, and consider the matroids $M_1|F$, $M_2|F$, $M_1.F$, and $M_2.F$.

Let $k = \min_{X \subseteq E} (r_1(\overline{X}) + r_2(\overline{X})\}$; we seek a common independent $k$-set in $M_1$ and $M_2$. If there is none, then $M_1 | F$ and $M_2|$ F have no common independent $k$set, and $M_1.F$ and $M_2.F$ have no common independent $k - 1$-set. The induction hypothesis and rank formula yield

$r_1(X) + r_2(F - X) \leq k - 1$ for some $X \subseteq F$, and

$r_1(Y + e) - 1 + r_2(F - Y + e) - 1 \leq k - 2$ for some $Y \subseteq F$.

We use $(F - Y) + e = \overline{Y}$ and $F - X = \overline{X + e}$ and sum the two inequalities:

$r_1(X) + r_2(\overline{X + e}) + r_1(Y + e) + r_2(\overline{Y}) \leq 2k - 1.$

Now we apply submodularity of $r_1$ to $X$ and $Y + e$ and submodularity of $r_2$ to $\overline{Y}$ and $\overline{X + e}$. For clarity, write $U = X + e$ and $V = Y + e$. Applying this to the preceding inequality yields

$r_1(X \cup V) + r_1(X \cap V) + r_2(\overline{Y} \cup \overline{U}) + r_2(\overline{Y} \cap \overline{U}) \leq 2k - 1.$

Since $\overline{Y} \cup \overline{U} = X \cup V$ and $\overline{Y} \cup \overline{U} = X \cap V$, the left side sums two instances of $r_1(Z) + r_2(\overline{Z})$, and the hypothesis $k \leq r_1(Z) + r_2(\overline{Z})$ for all $Z \subseteq E$ yields $2k \leq 2k - 1$. Hence $M_1$ and $M_2$ do have a common independent $k$-set.

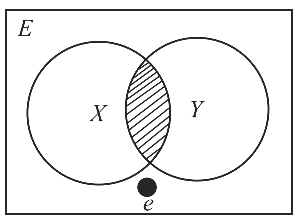



**Fig. 10.11: Intersection of Metroid**

## *Theorem 10.10: Metroid Union Theorem [1965]*

*If $M_1, \ldots, M_k$ are matroids on E with rank functions $r_1, \ldots, r_k$, then the union $M = M_1 \,U, \ldots, U\, M_k$ is a matroid with rank function $r(X) = \min_{Y \subseteq x}(|X - Y| + \Sigma r_i(Y))$.*

### *Proof*:

After proving the formula for the rank function, we will verify the submodularity property to prove that $M$ is a matroid. First we reduce the computation of the rank function to the computation of $r(E)$. In the restriction of the hereditary system $M$ to

the set $X$, we have $\mathbf{I}M \mid X = \{Y \subseteq X: Y \in \boldsymbol{I}_M\}$ and $r_{M\mid X}(Y) = r_M(Y)$ for $Y \subseteq X$. Thus $M \mid X = U_i(M_i \mid X)$, and applying the formula for the rank of the full union to $M \mid X$ yields $r_M(X)$.

Consider a $k$ by $\mid E \mid$ grid of elements $E'$ in which the $j$th column $E_j$ consists of $k$ copies of the element $e_j \in E$. We define two matroids $N_1$, $N_2$ on $E'$ such that the maximum size of a set independent in both $N_1$ and $N_2$ equals the maximum size of a set independent in $M$. We then compute $r_M(E)$ by applying the Matroid Intersection Theorem to $N_1$ and $N_2$. Let $M'_i$ be a copy of $M_i$ defined on the elements $E^i$ of row $i$ in $E'$. Lets $N_1$ be the direct sum matroid $M'_1 \oplus \ldots \oplus M'_k$, and let $N_2$ be the partition matroid uiduced on $E'$ by the column partition $\{E_j\}$.

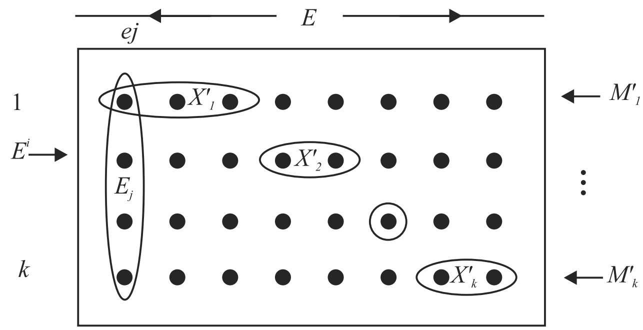



**Fig. 10.12: Metroid Union**

Each set $X \in \mathbf{I}_M$ has a decomposition as a disjoint union of subsets $X_i \in \mathbf{I}_i$, because $\mathbf{I}_i$ is a hereditary family. Given a decomposition $\{X_i\}$ of $X \in \mathbf{I}_M$, let $X'_i$; be the copy of $X_i$ in $E^i$. Since $\{X_i\}$ are disjoint, $\cup X'_j$ is independent in $N_2$, and $X_i \in \mathbf{I}_i$ implies that $\cup X'_i$ is also independent in $N_1$. From $X \in \mathbf{I}_M$, we have constructed $\cup X'_i$ of size $|X|$ in $\mathbf{I}_{N1} \cap \mathbf{I}_{N2}$. Conversely, any $X' \in \mathbf{I}_{N1}, \cap \mathbf{I}_{N2}$, corresponds to a decomposition of a set in $\mathbf{I}_M$ of size $|X'|$ when the sets $X' \cap E^i$ are transferred back to $E$, because $N_2$ forbids multiple copies of elements.

Hence $r(E) = \max\{| I | : I \in \mathbf{I}_{N1}, \cap \mathbf{I}_{N\,2},)$. To compute this, let the rank functions of $N_1$, $N_2$ be $q_1$, $q_2$, and let $r'_i$ be the rank function of the copy $M'_i$ of $M_i$, on $E_i$. We have $q_1(X') = \Sigma r'_i(X' \cap E^i)$, and $q_2(X')$ is the number of elements of E that have copies in $X'$. The Matroid Intersection Theorem yields $r(E) = \min_{X' \subseteq E'} \{q_1(X') + q_2(E' - X')\}$.

The minimum is achieved by a set $X'$ such that $E' - X'$ is closed in $N_2$. The closed sets in the partition matroid $N_2$ are the sets that contain all or none of the copies of each element–the unions of full columns of $E'$. Given $X'$ with $E' - X'$ closed in $N_2$, let $Y \subseteq E$ be the set of elements whose copies comprise $X'$. Then

$q_2(E' - X') = |E - Y|$, and $X'$ contains all copies of the elements of $Y$, so $q_1(X') = \Sigma r'_i(X' \cap E') = \Sigma r_i(Y)$. We conclude that $r(E) = \min_{y \subseteq E}\{|E - Y| + \Sigma r_i(Y)\}$.

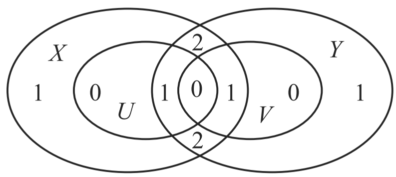
To show that $M$ is a matroid, we verify submodularity for $r$. Given $X, Y \subseteq E$, the formula for $r$ yields $U \subseteq X$ and $V \subseteq Y$ such that

$$r(X) = |X - U| + \Sigma r_i(U);$$
$$r(Y) = |Y - V| + \Sigma r_i(V).$$

Since $U \cap V \subseteq X \cap Y$ and $U \cup V \subseteq X \cup Y$, we also have

$$r(X \cap Y) \le |(X \cap Y) - (U \cap V)| + \Sigma r_i(U \cap V);$$
$$r(X \cup Y) \le |(X \cup Y) - (U \cup V)| + \Sigma r_i(U \cup V).$$

After applying the submodularity of each $r_i$ and the diagram below, these inequalities yield $r(X \cap Y) + r(X \cup Y) \le r(X) + r(Y)$.




$$|(X \cap Y) - (U \cap V)| + |X \cup Y) - (U \cup V)| = |X - U| + |Y - V| \qquad \blacksquare$$

## 10.14 Encodings of Graphs

Each model of encoding has assigning vectors to vertices, and the parameter is the minimum length of vectors that suffice. We study the maximum of this parameter over $n$-vertex graphs. The parameters are intersection number, product dimension and squashed-cube dimension.

An intersection representation of length $t$ assigns each vertex a 0, 1 vector of length $t$ such that $u \leftrightarrow v$ if and only if their vectors have a 1 in a common position. Equivalently, it assigns each $x \in V(G)$ a set $S_x \subseteq [t]$ such that $u \leftrightarrow v$ if and only if $S_u \cap S_v \ne Q$. The intersection number; $Q'(G)$ is the minimum length of an intersection representation of $G$.

The elements of $[t]$ in a representation correspond to complete subgraphs that cover $E(G)$. This motivates use of $Q'$ for intersection number: $Q(G)$ is the minimum number of cliques needed to cover $V(G)$.

A product representation of length t assigns the vertices distinct vectors of length t so that $u \leftrightarrow v$ if and only if their vectors differ in every position. The product dimension paradigm $G$ is the minimum length of such a representation of $G$.

By devoting one coordinate to each $e \in E(\overline{G})$, in which the vertices of $e$ have value 0 and other vertices have distinct positive value, we obtain pain $G \le e(\overline{G})$.

An equivalence on $G$ is a spanning subgraph of $G$ whose components are complete graphs.

A squashed-cube embedding of length $N$ is a map $f: v(G) \to S^N$ such that $d_G(u, v) = d_s(f(u), f(v))$. The squashed-cube dimension $q \dim G$ is the minimum length of such an embedding of $G$.

## 10.15 Ramanujan Graphs

Ramanujan Graph constitute a different family of regular graph. In contemporary mathematics, a great deal of interest has been shown in Ramanujan graph by research community in graph theory, number theory, cryptography and communication theory. A Ramanujan Graph is a $k$-regular connected graph $G$, $k \geq 2$ such that if $\lambda$ be any eigenvalue of G with $|\lambda| \neq k$, then $\lambda \leq 2\sqrt{k-1}$.

The following graph may be considered as Ramanujan graph:

(*i*) The complete graph $K_n$, $n \geq 3$.

(*ii*) Cycle $C_n$.

(*iii*) $K_{n,n}$, Here $k = n$ and $Sp\,(K_{n,n}) = \begin{pmatrix} n & 0 & -n \\ 1 & 2n-2 & 1 \end{pmatrix}$

(*iv*) The Petersen graph.

The Möebius ladder $M_h$ is the cubic graph obtained by joining the opposite vertices of the cycle $C_{2h}$. we have

$$\lambda_j = 2\cos\frac{\pi j}{h} + (-1)^j, \quad 0 \leq j \leq 2h-1$$

Take $h = 2p$ and $j = 4p - 2$. Then $\lambda_{4p-2} = 2\cos\frac{\pi(4p-2)}{2p} + 1 = 2\cos\frac{\pi}{p} + 1 >$ $2\sqrt{k-1}$ (when $p$ becomes large) $= 2\sqrt{2}$ (as $k = 3$). Hence, not every regular graph is a Ramanujan graph.

Let $G$ be a $k$-regular Ramanujan graph of order $n$, and $A$ is adjacency matrix. As A is symmetric, $\mathbb{R}^n$ has an orthonormal basis $\{u_1, \ldots, u_n\}$ of eigenvectors of A. Since $A_1 = k_1$, we can take $u_1 = \left(\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\right)^T$. Let Sp(G) = $\{\lambda_1 \geq \lambda_2 \geq, \ldots, \geq \lambda_n\}$, where $\lambda_1 = k$ and $Au_i = \lambda_i u_i$, $1 \leq i \leq n$. We can write A = $\sum_{i=1}^{n} \lambda_i u_i u_i^T$. (This is seen from the fact that the matrices on the two sides when postmultiplied by $u_j$, $1 \leq j < n$ both yield $\lambda_j u_j$.) More generally, as $\lambda_i^p$, $1 \leq i \leq n$ are the eigenvalues of $A^p$, we have

$$A^p = \sum_{i=1}^{n} \lambda_i^p u_i u_i^T \qquad \ldots (i)$$

Let $u_i = ((u_i)_1, (u_i)_2, \ldots, (u_i)_r, \ldots, (u_i)_n)$. Then the $(r, s)$th entry of $A_p = \sum_{i=1}^{n} \lambda_i^p (u_i)_r (u_i^T)_s = \sum_{i=1}^{n} \lambda_i^p (u_i)_r (u_i)_s := X_{rs}$. As $A$ is a binary matrix, $X_{r.s}$ is a nonnegative integer. Moreover, as $G$ is k-regular and connected, $\lambda_1 = k$. Set $\lambda(G) := \max_{|\lambda_i| \neq k} |\lambda_i|$. Then,

$$
\begin{aligned}
X_{rs} &= \sum_{i=1}^{n} \lambda_i^p (u_i)_r (u_i)_s \\
&= \left| \sum_{i=1}^{n} \lambda_i^p (u_i)_r (u_i)_s \right| \\
&\geq k^p (u_1)_r (u_1)_s - \left| \sum_{i=2}^{n} \lambda_i^p (u_i)_r (u_i)_s \right| \\
&= k^p \frac{1}{\sqrt{n}} \frac{1}{\sqrt{n}} - |\Sigma_0| \text{ where } \Sigma_0 = \sum_{i=2}^{n} \lambda_i^p (u_i)_r (u_i)_s \qquad \ldots (ii)
\end{aligned}
$$

Assume that $G$ is not bipartite, so that $\lambda_n = -k$. Hence, the eigenvalues $\lambda_2, \ldots, \lambda_n$ all satisfy $|\lambda_i| < \lambda(G)$, and therefore

$$
\begin{aligned}
|\Sigma_0| &\leq \lambda(G)^p \sum_{i=2}^{n} |(u_i)_r| |(u_i)_s| \\
&\leq \lambda(G)^p \left( \sum_{i=2}^{n} |(u_i)_r^2| \right)^{1/2} \left( \sum_{i=2}^{n} |(u_i)_s^2| \right)^{1/2}
\end{aligned}
$$

(by Cauchy-Schwarz inequality)

$$
\begin{aligned}
&\leq \lambda(G)^p \left[1 - (u_1)_r^2\right]^{1/2} \left[1 - (u_1)_s^2\right]^{1/2} \quad \text{(as the } u_i\text{'s are unit vectors)} \\
&\leq \lambda(G)^p \left[1 - \frac{1}{n}\right]^{1/2} \left[1 - \frac{1}{n}\right]^{1/2} \\
&\leq \lambda(G)^p \left[1 - \frac{1}{n}\right]
\end{aligned}
$$

Hence, the $(r, s)$th entry of $A^p$ is positive if $\frac{k^p}{n} > \lambda(G)^p \left(1 - \frac{1}{n}\right)$, that is, if $\frac{k^p}{\lambda(G)^p} > n - 1$.

on taking logarithms,

$$p > \frac{\log(n-1)}{\log(k / \lambda(G)} \quad (iii)$$

then every of $A^p$ is positive. Now the diameter of $G$ is the least positive integer $p$ for which $A^p$ is positive. Hence, the diameter $D$ of $G$ satisfies the inequality.

$$D \leq \frac{\log(n-1)}{\log(k / \lambda(G)} + 1 \quad (iv)$$

**Note:**

*The author has only touched upon the periphary of Ramanujan graph. A deeper study of Ramanujan graph requires expertise in number theory. Interested readers can refers to the two survey articles by Ram Murty and the relevent references contained therein. Some parts can also be seen in author's book entitled '**Elements of Number Theory and Cryptography**'.*

## EXERCISES

1. Prove that every tree of minimum order in which $G$ has a subtree representation is a clique tree of $G$.
2. Compute $\chi(G)$ and $\omega(G)$ for complement of odd cycle $\subset 2k + 1$.
3. Prove that every graph is the intersection graph of a family of subtree of some graph.
4. Determine the trees that split graphs, and construct a pain of nonisomorphic split graph with the same degree sequence.
5. Prove that every perfectly orderable graph is strongly perfect.
6. If $M$ is a matroid, then prove that the subbases we the spanning sets and the hypobases are the hypoplanes.
7. Characterize the graphs whose matchings from the family of independent sets of a matroid on the set of edges.
8. Prove that for $n \geq 4$, the minimum number of edges in a gossip scheme on $n$-vertices is $2n - 4$.

**9.** If $m = \binom{2k-1}{k}$, than prove that $K_{m, m}$ is not $k$-chorable.

**10.** Prove that $c¢_i(G) = c¢(G)$ for all $G$.

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